▲ Linking Representations Using Geogebra

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In secondary school mathematics, representing mathematical ideas and modelling situations generally takes the form of numeric, geometric, graphical, algebraic, pictorial, and concrete representation, as well as representation using dynamic software. Students should be able to go from one representation to another, recognize the connections between representations, and use the different representations appropriately and as needed to solve problems. (Ontario Ministry of Education, 2005, p. 16)
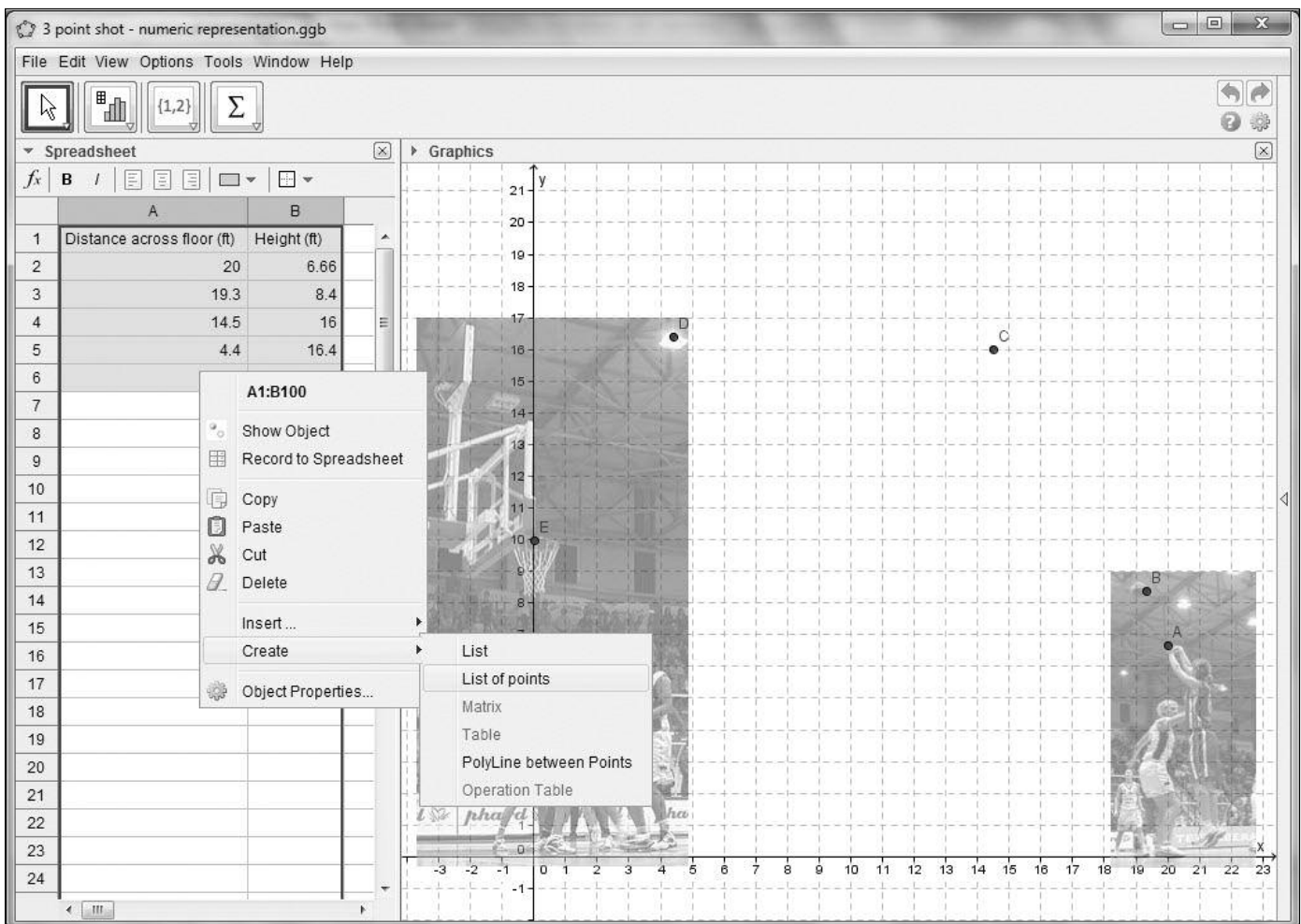

Figure 1: Pictorial & Numeric Representations of Ball Path **http://oame.on.ca/main/files/gazettelinks/Gz51i4RouletFigure1.html**

The quotation above from the Ontario Grades 9 and 10 curriculum document, and similar statements found in the guidelines for Grades 1 to 8 and 11 and 12, identify and expand upon Representing, one of the seven key mathematical processes. GeoGebra (International GeoGebra Institute, 2012a), an example of the "dynamic software" regularly referred to in our curriculum, can play a significant role in supporting students' learning to connect and move between different representations.

Giving an introduction to GeoGebra on paper presents a problem. In GeoGebra, the connections between representations are dynamic; that is, a change in one model (i.e., an algebraic equation) produces corresponding changes in other representations (i.e., a curve). Static images on paper cannot really illustrate this dynamic linking. Fortunately, GeoGebra's easy integration with the Web provides a way to expand the effectiveness of this article.

GeoGebra is written in Java and its **File > Export** option supports the creation of web pages holding applets, presenting the software in any state constructed by the user. A URL has been provided for each image in this article, and following these links will take the reader to active versions of GeoGebra as shown in the pictures. This sharing of dynamic tools is also supported by the fact that GeoGebra is free open-source software that you may download to your computer from **www.geogebra.org**. GeoGebra can operate within your web browser; thus, you can interact with the applets even if you do not want to install the program on your computer. This feature becomes very powerful when you wish to employ a course website to share GeoGebra learning objects with your class. Students can access the applets from any Internet-connected computer at home or in the school or local public library.

In this article, the topic of quadratic relations, found in both the Academic (MPM2D) and Applied (MFM2P) Grade 10 mathematics courses, is used as a context, but the tools and dynamic linking in GeoGebra serve for all secondary school courses and on into the college and university levels. For the elementary school grades, there is a reduced version, GeoGebraPrim (International GeoGebra Institute, 2012b); Prim for primary school, Grades 1–6 in Europe. The activity outlined below, the exploration of the path of a ball when a basketball player takes a jump shot, is not new, and has been conducted in a variety of ways with pencil and paper and supporting information and communication technology (ICT). GeoGebra provides some additional tools to enhance student exploration and aid the teacher's delivery of supporting materials via a course website.

Pictorial and Numeric Representations

Using a photograph (Finizio, 2005) of Italian national basketball team member, Sara Giauro, making a three-point shot and placing it on a grid, we can describe the location of the ball at a number of points as it travels from her hands to the basket. Starting the investigation with a photograph originally from a news report of a basketball game provides a real context for the problem, but also leads to some need to adjust the picture. For this application, the image, contained in a JPEG file obtained from the Web, has been split and mounted on the graph in two parts to compensate for the distance shortening produced by the photographer's telephoto lens. If the origin of a Cartesian plane is set on the floor directly below the basket, and the graph scale is adjusted to reflect the height of the basket rim at 10 feet, the location of the ball at five points along its flight path can be described by the numbers in the GeoGebra spreadsheet shown in Figure 1. Here, column A gives the distance in feet along the floor from Giauro's location, and column B records the height (feet) the ball is above the floor. GeoGebra provides many of the functions found in commercial spreadsheet software, but for this example, all we need to do is select the two columns and use the Create > List of points tool to plot the corresponding points on the graph. Students can, if they wish, rescale the graph and have the embedded photographs adjust by the same factors.

Algebraic and Graphical Representations

The locations of the five sample points and common knowledge would likely lead a class to suggest that the ball follows a parabolic path. If this investigation came somewhere near the end of the Quadratic Relations unit, students might suggest models of the forms $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$, with x representing the horizontal distance from Giauro in feet and y the height above the floor in feet. By entering, via the **Input Bar**, various examples of either equation, students could work toward values for the parameters. However, a better approach would be to use the **Slider** tool (Figure 2); create three adjustable parameters for a, h, and k; and then input the general function $f(x) = a(x - h)^2 + k$.

As students manipulate the values of the parameters—a, h, and k—the corresponding function expression is shown in the Algebra window, and the associated curve is displayed (Figure 3). With careful
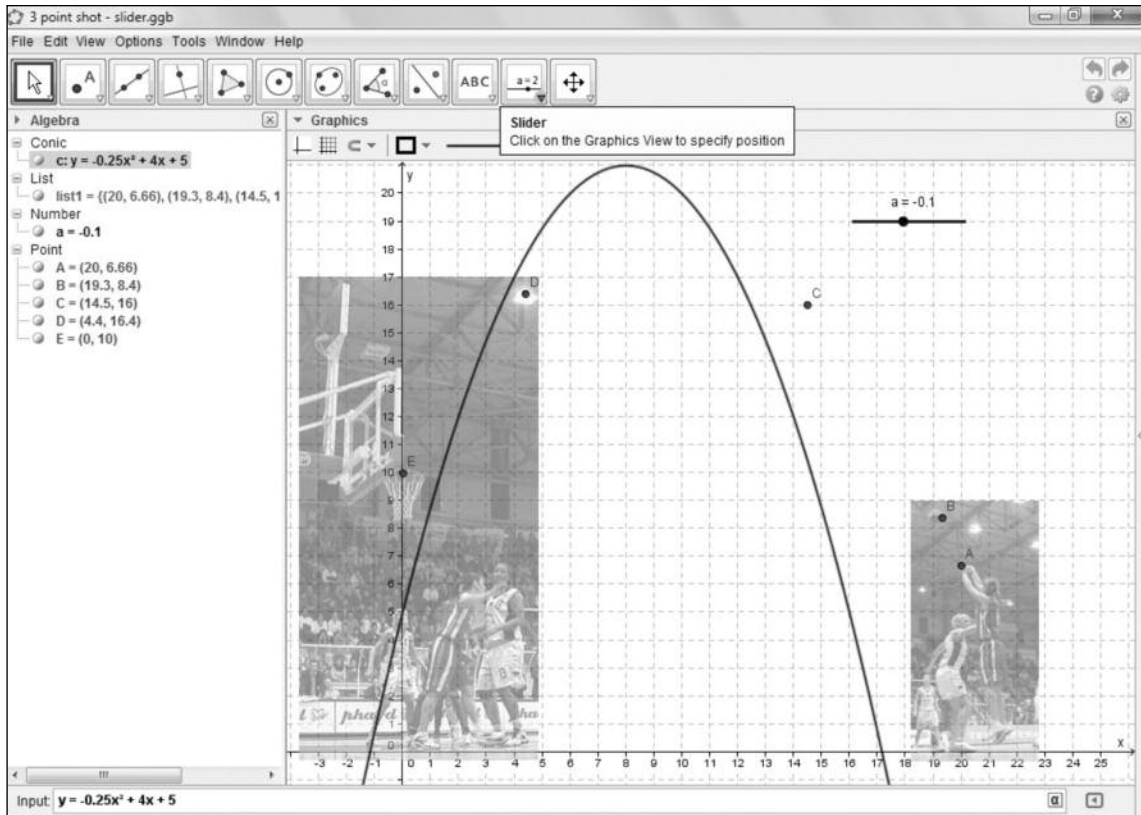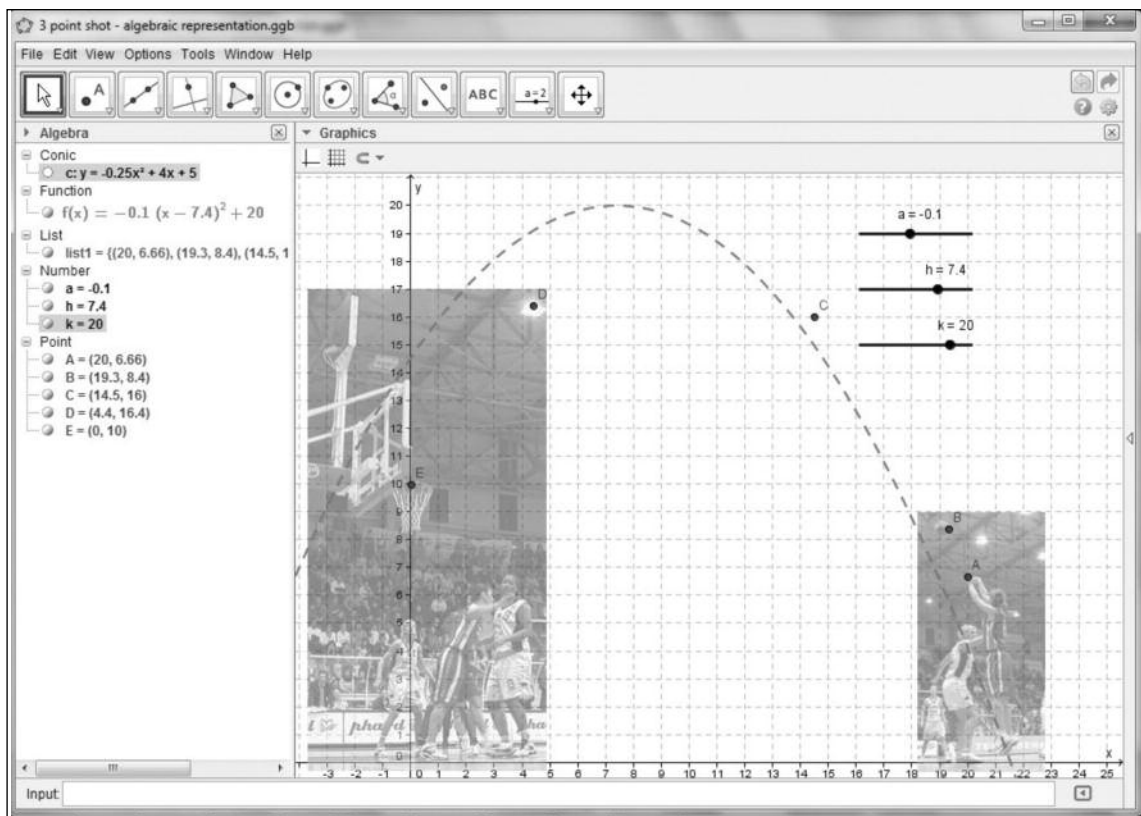
Figure 2: Trying a Quadratic Model URL: **http://oame.on.ca/main/files/gazettelinks/Gz51i4RouletFigure2.html**

Figure 3: Using Sliders to Explore a f(x) = a(x – h)² + k Model URL:
http://oame.on.ca/main/files/gazettelinks/Gz51i4RouletFigure3.html

adjustment of the parameter values, students can generate a curve that comes close to passing through the five sample points and develop an algebraic model that is something like $f(x) = -0.1(x - 9.25)^2 + 18.75$.

Using the Algebraic Model in the CAS Window

In the photograph, Giauro has driven to the corner and is shooting her three-pointer across the court. Suppose Giauro had made a shot with about the same arc, but had aimed too high. That is, suppose the algebraic model for her shot were $f(x) = -0.1(x - 8)^2 + 21$. In this case, the ball would pass over the basket and could be caught by another player on the opposite side of the court (see Figure 4). If you were on the court and wished to catch the ball just above your head at 5.75 feet, how far away from the basket should you stand? Problems such as this can be solved by manipulating the expressions of a model in the Computer Algebra System (CAS) window of GeoGebra (Figure 4). Copy-and-paste

brings the algebraic model to the CAS window, and substituting 5.75 for $f(x)$ gives the quadratic equation, $5.75 = -0.1(x - 8)^2 + 21$, to which we can apply the Solve Numerically tool (see Figure 4). The negative root tells us that a player wishing to catch the ball at a point 5.75 feet above the floor must stand 4.35 feet away from the basket across the court from Giauro.

Geometric Representation

The Euclidean loci descriptions of the conics, such as the focus-directrix definition of a parabola, are no longer topics of the Ontario mathematics program, but the Graphics (geometry) view of GeoGebra could be used to briefly introduce the concept via an online learning object. In Figure 5, a free point labelled **Focus**, and a second point **D**, on the y-axis, have been added to the coordinate plane. With a line constructed perpendicular to the y-axis through point D as the directrix, we can use the **Conic** > **Parabola** tool to produce a curve that students could explore by dragging the points, Focus
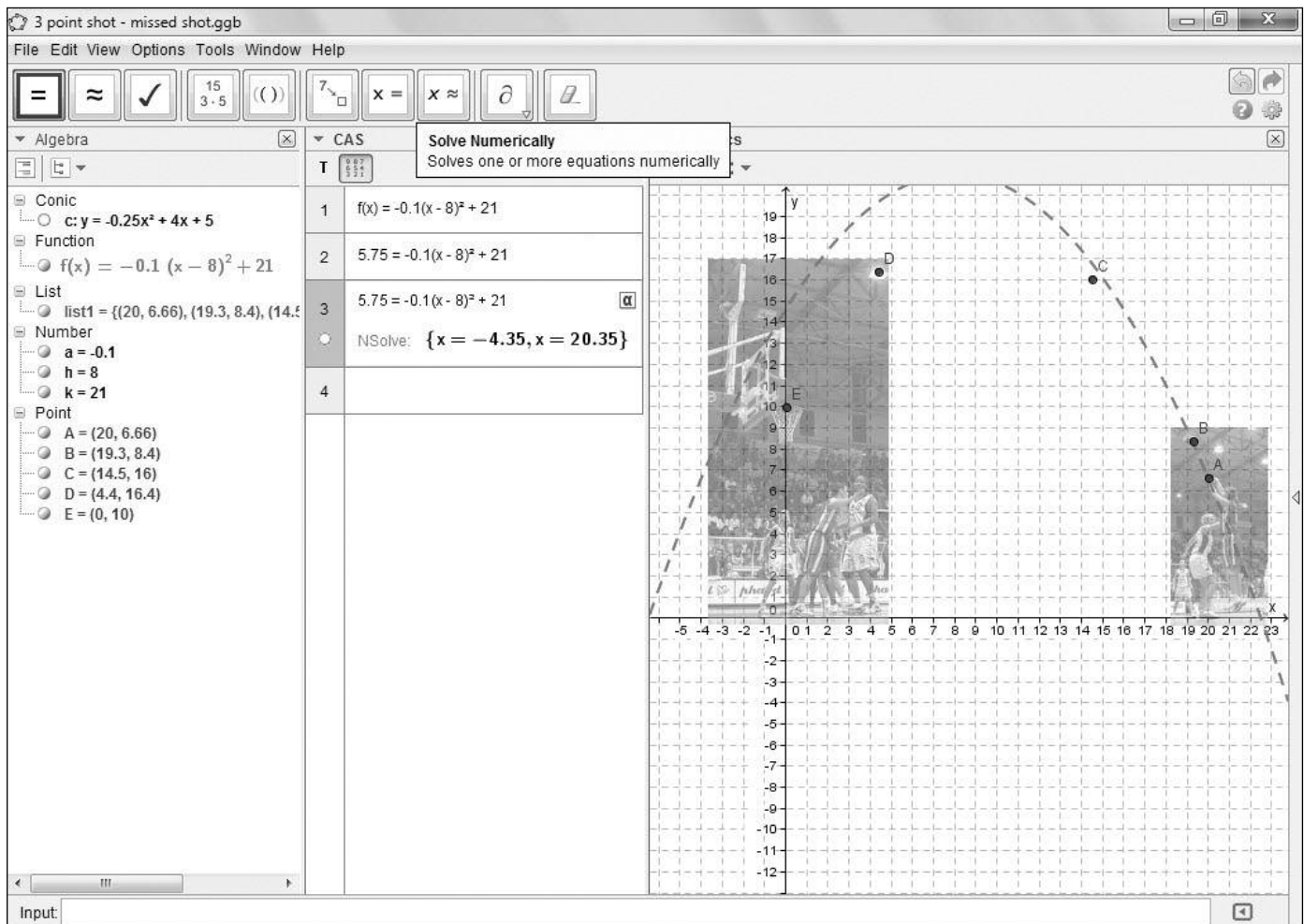

Figure 4: Using the CAS to Solve a Problem
URL: **http://oame.on.ca/main/files/gazettelinks/Gz51i4RouletFigure4.html**

and D. To make the activity more engaging, a moving image of a ball can be attached to the curve, and now the students' task is to develop a curve that takes the ball from Sara's hands to the basket. Using the **File** > **Export** > **Dynamic worksheet** option, a web page (see Figure 5) holding this activity could be generated and mounted on a course website.

Statistical Modelling

The above approach linking numeric, graphical, algebraic, and geometric representations is effective with classes initially exploring the connections between parabolic curves and quadratic functions. Once understanding has been developed, a class might move on to using the statistical modelling tools of GeoGebra to build quadratic algebraic representations needed for problem solving. The initial Spreadsheet view of the five sample points along the path of the ball also provides statistical tools, such as Two Variable Regression Analysis (see Figure 6). Clicking on this tool and

selecting a Polynomial model of degree 2 brings up a window that shows a regression model, $y = -0.1x^2 + 1.93x + 9.98$ (see Figure 6). This algebraic representation could then be copied over to the Algebraic and Graphic views of GeoGebra and employed in the CAS view to solve problems such as the player position above.

GeoGebra Collaboration

With GeoGebra available in 62 languages, the software is developing a large worldwide community that comes together on the GeoGebra website (**www.geogebra.org/cms/en/community-info**) to provide mutual advice and support, and share many GeoGebra-based online learning tools (**www.geogebra tube.org**). Visit these sites if you wish more information, and follow the URLs presented with each figure to explore in more detail the steps in the constructions above. In addition, the ability to export GeoGebra applets and embed them in wiki and course-learning
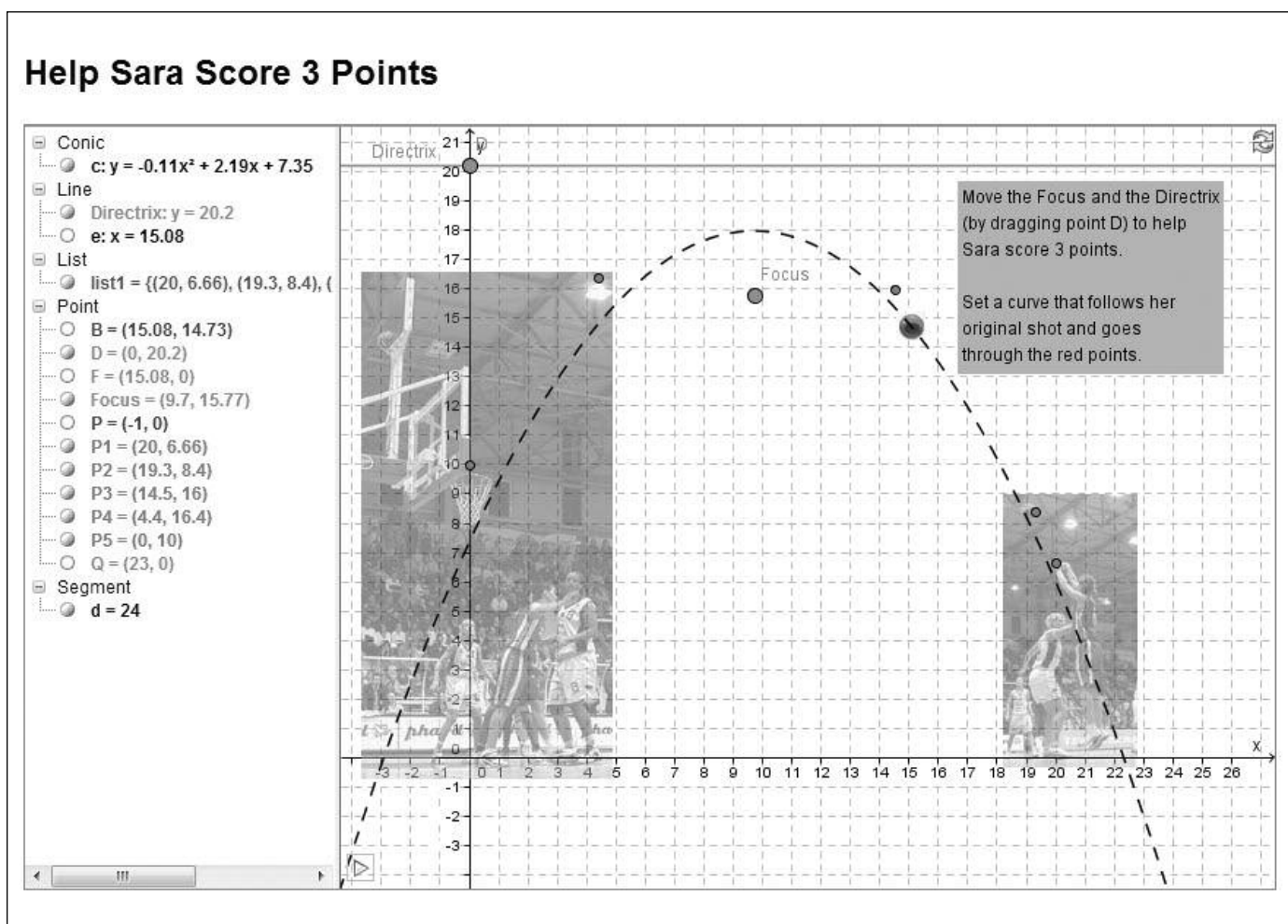

Help Sara Score 3 Points

Figure 5: An Online Learning Object for Exploring the Focus-Directrix Parabola Definition
URL: **http://oame.on.ca/main/files/gazettelinks/Gz51i4RouletFigure5.html**

management system (LMS) pages can aid in the development of a classroom mathematics learning community. Students can post their GeoGebra work along with explanations of their thinking, and others in the class can interact with the applets and comment on the work (Lazarus & Roulet, 2013, this issue).

References

Finizio, M. (Photographer). *Three point shoot by Sara Giauro (Phard Vomero Napoli) during FIBA Europe Cup Women Finals 2005.* (2005). [Photograph]. Naples, Italy: Creative Commons (cc-by-sa-2.0-it). Retrieved from **http://en.wikipedia.org/wiki/File:Three_point_shoot.JPG**

International GeoGebra Institute. (2012a). *GeoGebra* (Version 4.2.4.0) [Software]. Available from **www.geogebra.org/**

International GeoGebra Institute. (2012b). *GeoGebraPrim* (Version 4.0.41.0) [Software]. Available from **www.geogebra.org/webstart/4.0/GeoGebraPrim.jnlp**

Lazarus, J., & Roulet, G. (2013). Communication in a blended math-talk learning community: Extending classroom boundaries. *Gazette, 51*(4), 34-40

Lazarus, J., & Routlet, G. (2013, March 13). Supporting Collaborative Mathematics Using a Wiki, GeoGebra, and Jing. *In Math Future*. Retrieved from **http://mathfuture.sikispaces.com/SupportingCollaborativeMath**

Ontario Ministry of Education. (2005). The Ontario curriculum, Grades 9 and 10: *Mathematics, 2005* (revised). Toronto: Queen's Printer for Ontario. Retrieved from **www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf**

Note 1: Resources to assist you and your students while working with GeoGebra can be found at **http://wiki.geogebra.org/en/** and **http://queensgeogebra.pbworks.com**.

Note 2: Using the URLs listed with the figures in a web browser or clicking on the links embedded in the PDF version of this article should automatically load the GeoGebra applet. If this does not happen on your computer you may need to go to the GeoGebra download site at: **http://www.geogebra.org/cms/download**, and download and run the Java Webstart program. ▲
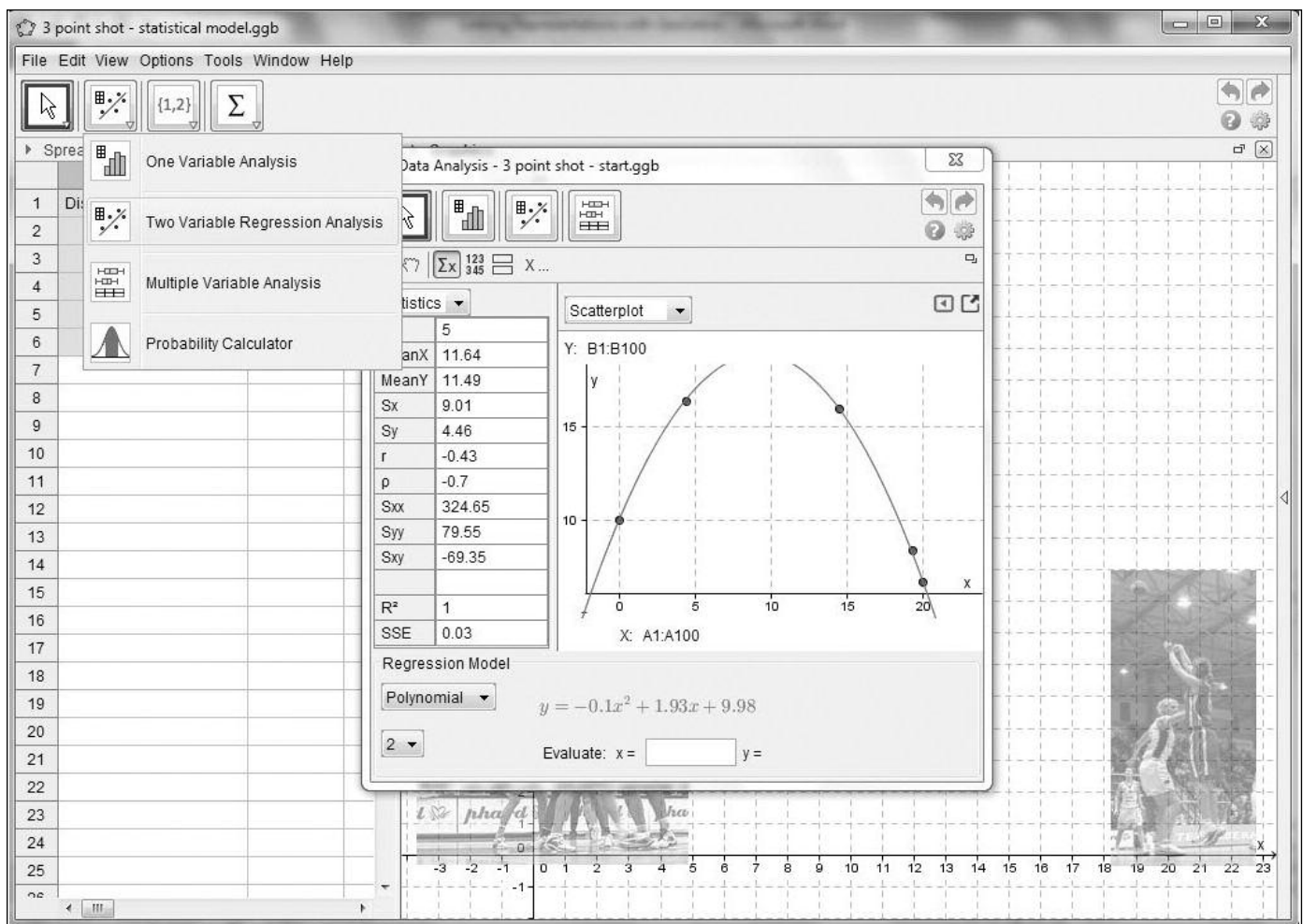

Figure 6: Using Two Variable Regression to Generate an Algebraic Representation
URL: **http://oame.on.ca/main/files/gazettelinks/Gz51i4RouletFigure6.html**