Ontario Mathematics Gazette

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OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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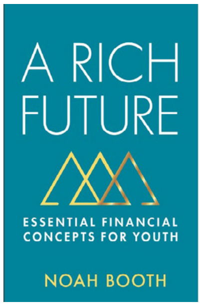

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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line spacing (not double). Articles should not exceed five numbered pages of text, and figures, images, and photographs should be placed in the text close to where they belong, with captions. The photographer's permission is required, and for photos of students under the age of 18, the written permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or declined. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

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The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

▲ Editor's Report

TIM SIBBALD, OCT, PH.D.
gazette@oame.on.ca

Tim Sibbald is a professor in the Schulich School of Education, Nipissing University, with a focus that includes mathematics education. He is a former President of OAME/AOEM and has held a variety of other OAME/AOEM posts over the years.

The International Congress of Mathematicians (ICM) is a global math conference that occurs every four years. It is run by the International Math Union (IMU), where the members are countries and, yes, Canada is a long-standing member. While the first ICM was in Zurich, Switzerland in 1897, the first official ICM in North America was in Toronto in 1924—one century ago. The Toronto ICM was organized by John Fields, after whom the Fields Medal—the most prestigious award in mathematics—is named. The Fields Institute recently mentioned this event, and it caused a moment of thinking about growth, which Sandra Jean talks about in her "President's Message."

Within the IMU, the ICM is for mathematicians, and because of its size, a separate global conference focuses on mathematics education. The mathematics education focus falls under ICME, and it is ICME that worked to have pi-day, March 14, recognized by UNESCO as the International Day of Mathematics. But I digress….

Years ago I read a *Gazette* Editorial of Dr. Immaculate Kizito Namukasa, who recounted attending ICME in South Korea with 3600 delegates. I recall thinking I would like to see what that is and to have an experience like Immaculate had. Although, I managed it during the pandemic and attended ICME in Shanghai, China. Well, I wasn't actually in Shanghai; I was attending virtually. It was an experience, despite the pandemic, because it ran between 4 a.m. and 11 a.m. for attendees in our time zone. And now, the next ICME is occurring in Sydney, Australia, and I will be there in person.

Let me go off on a pragmatic tangent for a moment. The preparation of the September issue coincides with my being in Australia. As members of the Board of Directors (see Figure 1) know, I have joked about various countries not having the Internet when I want to avoid meeting during travel. No surprise that Australian Internet won't be working while I am there… and that means that Ralph Connelly will be the guest Editor for the September issue of the *Gazette*. Ralph has been the Editor twice before and knows what he

Figure 1: *The March Board of Directors virtual meeting, where everyone received cookies (Photo and cookie credit: Sandra Jean Price)*

is doing. He has always been supportive behind the scenes, and I am delighted to be able to step away for an issue, knowing he will fill the Editor role.

With Sandra Jean reflecting on the learning she experienced during her tenure as President and with the Board of Directors (which thankfully will continue), I was led to reflect on my own learning, particularly when I considered how to convey the role of ICME in the larger picture of mathematics education. What is its role in mathematics education?

Teachers are well versed in working with policies, such as the curriculum, and enacting it in interesting and inventive ways in the classroom. The *Gazette* is a forum for sharing those ideas, and if you are reading this in the summer, take a moment to think if there is some nugget of an idea that you should perhaps write up to share through the *Gazette*. Some teachers engage beyond policies, with initiatives to develop updates to policies or to inform other teachers of good, often better, practices (I am loathe to call them “best practices”).

The same way that teachers bridge between policies and practice, members of the Ontario Ministry of Education, school boards, and authorities guide the process of developing new policies (our own Anne Yeager was involved in developing several math curriculum documents), and they bridge research in that process. Often it is well-established research that has broad support, as opposed to new or exploratory research.

The research element is the focus of university professors. It is not exclusionary and often engages some teachers, teacher candidates, and graduate students. However, the timing and cost of research conferences is often contrary to teachers being engaged. The research element provides a space for discussion of exploratory research, ideas, conceptual models within education, and a

Who Is OCMC?

“We would like to formally challenge your math club to a duel. Accept my proposition and reply to this email. Otherwise, I’ll assume that you surrendered.” This was the beginning of the Ontario Competitive Mathematics Committee (OCMC) in November 2022. The OCMC is one of Ontario’s biggest student-run, formally registered non-profit organizations dedicated to encouraging interest in mathematics and analytical thinking. Not only do we host high school math contests, but we also provide high school students with leadership opportunities as proctors and connect math lovers across schools. Though headquartered in Oakville, Ontario, we are open to helping everyone who could benefit from our contests, and we have already attracted participants throughout the province and even internationally in Dubai.

Every year, we host multiple contests: the Ontario Mathematics Competition (October), the Ontario Invitational Mathematics Examination (December), and the Tesseract Mathematics Challenge (March). All of our contests are free, and we provide monetary and merchandise prizes for top-performing contestants. For more details and past contest papers, please visit our website (www.ontarioocmc.ca). If your school or math club might be interested in participating in our contests, we would love to hear from you by email at general@ontariocmc.ca or through Instagram @ontariocmc myriad of notions that reflect ways one can think about education. I recently had a research conversation, where I questioned the very nature of curriculum documents—the details are not germane to this editorial, but research provides different perspectives that can inform policy development and everything else in education.

In case you wonder, my specific focus at the ICME Conference has to do with the development of a modern theoretical framework for interdisciplinary (including STEM) education. If memory serves me correctly, there are four researchers from four countries presenting on the topic. Fundamentally, the issue is that the last comprehensive framework was developed in 1948, oriented to university departments collaborating, and was devoid of any technology considerations. You might wonder how policies around interdisciplinary teaching (including STEM teaching) can be made with such an outdated and misaligned framework, and therein is the reason why some researchers have been addressing the concern.

News

Esther Mahlangu was awarded an honorary doctorate by the University of South Africa in recognition that her artwork "is a form of mathematics," "contemporary paintings that are colourful, geometric, and reference her Ndebele heritage." (See www.iol.co.za/news/dr-esther-mahlangu-to-get-an-honorary-doctorate-from-unisa-4d1eee48-e7ff-4470-abcc-53752dd25692 for details.)

Kyne Santos, a Canadian drag queen, authored *Math in Drag*, which was published in March by Hopkins University Press. **Contact the Editor if you would like to earn a free copy by writing a book review.**

In This Issue…

Sandra Jean Price brings her final "President's Message," recognizing the transition. We have an obituary of the first elected OAME President, **David Alexander**, in **Peter Saarimaki**'s words "a life well lived." **Pamela Brittain** and **Heather Theijsmeijer** highlight the various events at the Fields Institute for Research in Mathematical Sciences and the Ontario Mathematics Coordinators Association, respectively. **Jacqueline Hill** makes an NCTM connection by talking about **Don Fraser**.

By way of columns, **Jennifer Holm** considers different subtraction processes for Primary and Junior grades. **Shawn Godin** continues his series of columns that have explicitly identified math processes in action—I think it has been a fascinating series, and I hope you find it insightful.

Figure 2: *Esther Mahlangu at an exhibit of her art (Photo from www.vmfa.museum/connect/esther-mahlangu-vmfa/)*

Iain Broadie and **Beyza Sezer** interview **Adam Leyland** with a connection to **OAME Talks**. **Lynda Colgan** speaks up for gifted and talented students, who often seem to have their enrichment needs overlooked. **Carly Ziniuk** brings out-of-this-world resources to our attention. **Jeff Irvine** draws attention to achievement goal theory.

The issue also includes an article by **Ann Kajander** and **Walid Khneisser** about assessment in destreamed Grade 9.

Thank Yous

After the last issue, I received an email stating that it was the best issue yet. This came on the heels of the individual having said that about the previous issue. I might like to take some credit, but truly it is a team effort. Quality comes from contributions by writers and columnists. Those contributions are wordsmithed by Associate Editors, **Anne Yeager** and **Peter Saarimaki**, who have more years of expertise than I can publicly specify. Polishing is the art and craft of **Gitta Berg**, who astounds with the acuity of her eye. But even the loveliest of gemstones can be overlooked if it is not set just right, and that is what **Penny Clemens** does every time. I would also like to add **Robert Sherk**, our Advertising Manager, and **Pole Printing**, who make and mail the hard copies that continue to be enjoyed by quite a few members and used in libraries in several countries.

With a stellar editorial team, I was comfortable that the September issue could be handled while I travel in Australia. However, after getting caught by two-factor authentication and other security measures, I decided it was going to be more reliable to step away for an issue. I am thankful that **Ralph Connelly** has agreed to step into the role. His support over the years, in very concrete terms, has demonstrated a level of dedication that can only be applauded. Thank you, Ralph! ▲

▲ President's Message

SANDRA JEAN PRICE
sandra.jean.price@oame.on.ca

Sandra Jean has been on the Board of Directors of OAME/AOEM for almost 20 years. She is currently teaching in the Durham District School Board and has more than 25 years of teaching experience. She has had many roles, including classroom teacher, Math Coach, and Instructional Leader of online Math Additional Basic Qualifications (ABQs). Sandra Jean has contributed to a variety of Ministry projects and was a coordinator of the most recent OAME/AEMO writing projects. Sandra was the recipient of the 2008 OAME Award for Exceptional and Creative Teaching in Elementary Mathematics for her innovative instructional practices, integration of technology, and belief in success for all.

As the school year is nearing its end, so is my term as OAME/AOEM President. Over the last two years, much was learned from my fellow Executive and Board members, resulting in professional growth. Now is an opportunity to reflect on this journey from my first years on the Board and through the last two years.

I first became a Chapter Representative for Pine Ridge Math Association (PRMA) in the Fall of 2003 after recently joining OAME/AOEM for the sole purpose of being a Committee Chair on the 2003 conference-organizing committee. That first conference experience, working alongside many colleagues and even some former teachers, shed some light on the many supports that OAME/AOEM offers. Jacqueline Hill was the chapter's President at that time, and she gave me a bit of a nudge, as she thought that I would enjoy being on the Board of Directors of OAME/AOEM representing our chapter. I attended my first Board meeting with great trepidation, and I may not have spoken to many people aside from Jacqueline during that first year. But Jacqueline, as always, was correct. Sharing ideas with other like-minded individuals with a similar passion for mathematics education was energizing and inspiring. A spark was ignited for me, one that would feed my enthusiasm for mathematics, teaching, and learning.

Bill Otto, Chris Suurtamm, Kathy Kubota, Pat Margerm, Mary Bourassa, Judy Dussiaume, and Heather Boychuk were among the many Board members during that time frame. Although I was quiet and in complete awe of the expertise and talent of the people in the room, these individuals all welcomed me to the Board. I'm not sure if any of them truly know how much of an impact they had during my early time on the Board. If nothing else, hopefully they know that I kept returning year after year to keep working with them and all the other incredible individuals who have contributed to the Board over the years.

In more recent years, there have been others who have provided guidance and encouragement. Greg Clark, our long-time Webmaster and Technology Coordinator, has been on the Board during my entire tenure. He chairs the Chapter Rep committee and is integral to keeping us moving forward, especially in terms of technology. Fred and Lynda Ferneyhough, our amazing Executive Directors, steer OAME/AOEM in the right direction, and continuously support all of us and our work to help Ontario teachers of mathematics. They are the unsung heroes of our organization, which could not run without the leadership and guidance that they provide daily.

Paul Alves and Judy Mendaglio provided ongoing encouragement and support when I joined the Executive Committee as a Vice-President. As President, both were willing to do some extra reading and have deep, meaningful conversations to work through ideas. Not only are they wonderful educators and leaders, but they are also valued friends.

This year, it has been wonderful serving on the Executive with Lynda and Fred Ferneyhough, Paul Alves, Kit Luce, Melissa Black, and Bart Vanslack. In September, Bart Vanslack will be leaving the Board, as his term in the role of Vice-President ends, and Mark Chubb will be joining the Executive Committee as a Vice-President. Kit Luce will be the new President of OAME/AOEM as of September 1. Over the summer, our new Executive for the 2024–2025 year will be looking ahead and continuing to plan amazing events, including a virtual Leadership Event in the Fall.

Others who must be acknowledged include Timothy Sibbald, the *Gazette* Editor, Marc Husband and Tina Rapke, our *Abacus* Co-Editors, and Robert Sherk, the *Gazette* Advertising Manager. They, along with the editorial volunteers, have done an excellent job putting OAME/AOEM information out into the world. In addition, the contributions of David Petro, producer of our OAME Talks podcasts and webinars, and Iain Brodie and Hatice Beyza Sezer, co-hosts of the Coding in the Classroom podcasts, are sincerely appreciated.

Thank you to the organizing committee of the OAME 2024 Annual Conference. The O34ME (Ottawa Zone for Mathematics Education) and QSLMA (Quinte St. Lawrence Mathematics Association) chapters collaborated to host an incredible conference in May. As always, the conference allows all of us to attend professional learning in a fashion that gives us choice. After all, each of us has different strengths and areas for growth, and we are at different places on the continuum of learning. This conference allows for the self-direction of our own learning to meet each person's needs. Making this happen requires many volunteers and three years of planning behind the scenes. Thank you to all the volunteers for their time and effort.

To all of you, the members of OAME/AOEM and our local chapter organizations, thank you for continuing to learn, collaborating and working to serve the students of Ontario. Thank you for igniting a spark in teachers and encouraging them to share ideas for the first time, present at a conference for the first time, or even join the Board of Directors for the first time. You are the reason that our organization exists and why many of us continue to want to be part of it.

As my final message as President comes to a close, I am hopeful that all of you have had a wonderful school year and are able to take the time to relax and recharge this summer. Whether you are new to the profession, already enjoying retirement, or are currently somewhere along the journey in between, I hope that mathematics and learning continue to ignite sparks for you every day. ▲

David William Alexander: A Look at a Life Well Lived

March 18, 1935 – January 30, 2024

BY PETER SAARIMAKI
OAME PAST PRESIDENT AND LIFE MEMBER
OMCA PAST PRESIDENT AND LIFE MEMBER

On January 30, 2024, OAME/AOEM and mathematics education in Ontario and beyond lost a true pioneer.

David Alexander, or Dave to all who met him, was OAME's first President (1973–1974), having previously been President of the Ontario Association of Teachers of Mathematics (OATM, 1969–1970). The OATM was one of the two math organizations that came together to form OAME, the other being the Ontario Mathematics Commission (OMC).

Dave's role in this amalgamation is described in the Centennial Edition of the *Gazette* (April 1991, 29(3), p. 7):

> In 1969,… the President of OMC approached OATM with the news that financial problems might cause them to disband. [OATM] President Dave Alexander expressed the concern that… less OATM council meeting time had been spent on educational issues.

To find a solution to both these issues, Dave then became part of the joint committee that came together in 1970, and worked through several annual conferences to finally make OAME official in 1973 in North Bay, where Dave was elected as our first President. Tom Griffiths, OAME President 1976–1977, remembers working with Dave as an OATM councillor on the transfer to OAME, saying "he was a very special person."

I knew of and worked with Dave for well over 50 years. His influence, while often quiet and subdued, was extensive.

So how did it all start? For early schooling, Dave attended Bedford Park Public School and Lawrence Park Collegiate Institute in Toronto. He then earned his post-secondary education at the University of Toronto and at the

State University of New York in Buffalo. After those experiences, he went on to teach at Bloor Collegiate Institute (Toronto), the University of Toronto Schools (UTS), and the Faculty of Education at the University of Toronto (FEUT).

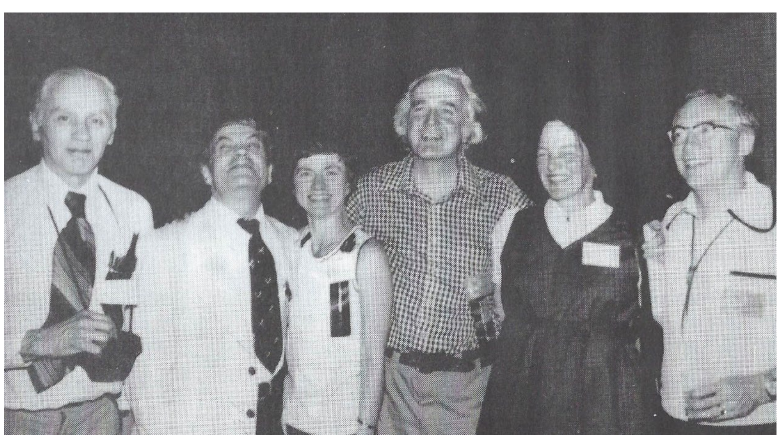
In September 1967, the first Computer Science course in Ontario was authorized, and a pilot course was started at Malvern Collegiate Institute, but there weren't any resources, except—Dave had taught an experimental course in computer training in 1965–1966 at UTS, and then written two articles in the *Gazette* (March 1967, 5(2)): "Computer Courses in Secondary Schools" and "Introduction to Computing." Talk about timing!

During these early years, Dave also took a very active role in professional organizations and mathematics curriculum development. He became the chair of the Mathematics Education Department at FEUT. He worked and collaborated with such early provincial math leaders as J.T. Crawford, Samuel Beatty, D. Mumford, and others.

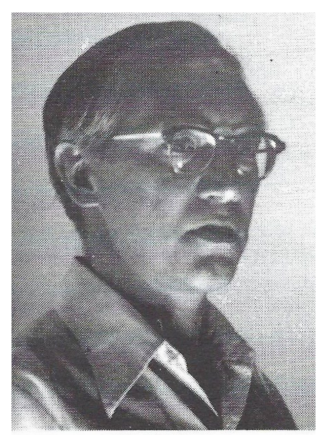
Conferences seemed to be of high interest to Dave, as they brought together the best in math education for the benefit of everyone. Dino Dottori, OAME President 1986–1987, recalls:

> I remember Dave well, and working with him when we moved the annual conference out of OCE (with lunch at the United Church across the street) and into the Skyline Hotel. Dave chaired and Father Paul and I filled two ballrooms with exhibitors. So now, as we mourn the loss of a colleague, we look back and remember the good times.

Dave was also very active in NCTM. When they held a regional conference in Toronto in the Fall of 1973, Dave was the chair of the Local Arrangements Committee (LAC), and I was chair of Student Volunteers—our first personal connection. He made sure everyone knew what their responsibilities were, and also ensured we felt part of the larger team.



Left to Right: *Paul Sherk, John del Grande, Lorna Morrow, John Egsgard, Lynn Egsgard, and Dave Alexander*



FROM THE GALLERY: DAVE ALEXANDER

APRIL 1991 ▲ 19

In the early 1980's, Dave went on secondment for two years as an Educational Officer with the Ministry of Education (MoE). In that capacity, he worked on a teacher in-service program for calculus. He also formed a special committee to work on the mathematics curriculum for Grades 7 to 13. At the last minute, Education Minister Bette Stephenson decided that students should complete the high school mathematics curriculum in four years, not five, and thus, the last courses were designated as OACs (Ontario Academic Credits). Dave and others were not thrilled with the sudden shift, and as Tom Griffiths said, "We felt really sorry for David at the time." To counter these events, the University of Waterloo awarded Dave with a Descartes medal for his contributions to the "noble cause of mathematics and computer science education."

NCTM held their only annual conference outside of the United States in Toronto in April of 1982, and Dave again was the LAC chair, while I was now chair of Hospitality for the delegates. This involved several years of planning and meetings for this major event, and Dave was very good at juggling many balls at once. And it was a family affair, as Bonnie, his wife and partner, was in charge of the hospitality for spouses of the NCTM executive and keynote speakers.

Even though he was no longer at the MoE, Dave was still influencing math education in Ontario as the curriculum leader for a provincial review of mathematics for the Grade 10 General and Grade 12 Advanced courses from 1989–1992. As Don Fraser said, "Dave is a conscientious, hard worker. Given a job, he does it 100%" (*Ontario Mathematics Gazette*, 33(2), p. 3).

In the early 1990's, beyond Ontario, Dave was the local organizer for a working group on Methods of Implementing Curriculum Change for the International Congress of Mathematical Education (ICME), and an author for the Council of Ministers of Education, Canada on their report on Mathematics Assessment in 1993. He also authored several math textbooks.

Then, as Dave neared retirement, he and Bonnie accepted the role of Executive Directors for OAME/AOEM, as Don and Carol Attridge finished their time in the role. As a Councillor and later Director for OAME/AOEM, I personally saw how they worked so well together. While Dave would worry over budgets and meeting arrangements, Bonnie would keep track of the minutes and how the meetings were running. There would be weekly meetings or phone calls with Presidents and the Executive, processing the paperwork of the organization, planning for meetings or conferences, overseeing the production and distribution of the *Gazette*, and so on.

Dave and Bonnie were Executive Directors from 1994 to 2001, when again, Dave decided to retire, this time to their new home on Lake Rosseau. At this point, I was the Past President of OAME/AOEM and thus responsible for finding replacements for Bonnie and Dave. During this time, Dave was most helpful in discussing all the details involved in the role, how jobs could be shared, and what kind of storage space would be needed (for all of OAME's publications, brochures, notepads, past editions of the *Gazette*, etc.).

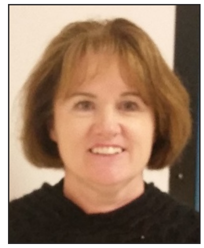
To share all the Executive Director details, Bonnie and Dave invited Janice and me to Lake Rosseau for a weekend, where Dave's interest in sailing, golf, and gardening was very evident. Even after retiring now several times, Dave still contributed to OAME/AOEM, including articles in the 40th-Anniversary Edition of the *Gazette* (June 2012, 50(4)), and more recent 50/60/50 Anniversaries Edition (March 2022, 60(3)).

In retirement, Dave and Bonnie spent many winters in Florida, until Dave started to exhibit early stages of dementia. Bonnie was able to care for Dave at home with home-care help for several years till late last Fall, when his condition deteriorated very quickly.

Ron Lancaster summed up our thoughts very well:

> I remember him very well and always held him in high regard. He was a one-of-a-kind leader in mathematics education. When I compare David to some of the leaders/influencers today, the differences that I see are that he was not in it for the money, he was not into self-promotion, and he truly loved mathematics and working with people. ▲

▲ OAME/NCTM Report: Canadians Give True Contributions to NCTM!



JACQUELINE HILL
jacqueline.hill@bell.net

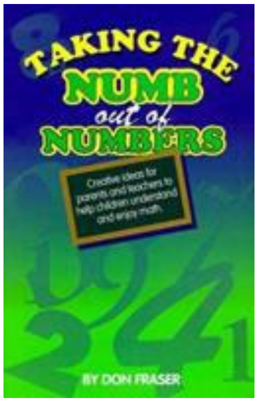
Jacqueline Hill is a Grades 9–12 teacher of mathematics. She is a Past President of OAME and OMCA, as well as the recipient of the Award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette *for a number of years.*

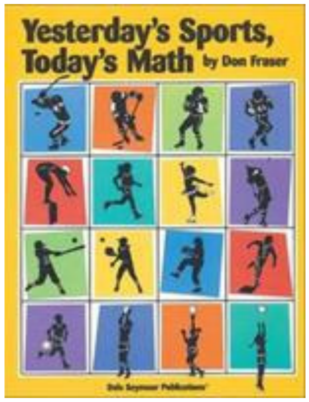
If anyone still has sore arms from busily acting like a hummingbird flapping its wings, or is looking for their borrowed wristwatch, then you are in good company! Don Fraser has used humour and fun to pique the curious minds of teachers and students for the past 50 years. He has presented at NCTM conferences in Canada and the United States. In the United States, Don has spoken from Alaska to Florida and Oregon to New York. Don has also spoken multiple times in every province, except Prince Edward Island.

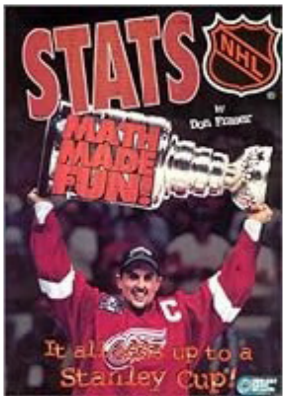
A highlight for Don was being the banquet speaker at the NCTM National Conference in Seattle, Washington, in 1992. He has done many workshops and keynotes for both NCTM and OAME. Several times, Don has been the closing speaker. He loves to refer to his closing spot as "C—LOSER." This always breaks the ice and gets a laugh. He smiles when he recalls the occasional packed rooms and auditoriums—the times when people lined up outside the hall, and those who showed up an hour early just to hear him speak.

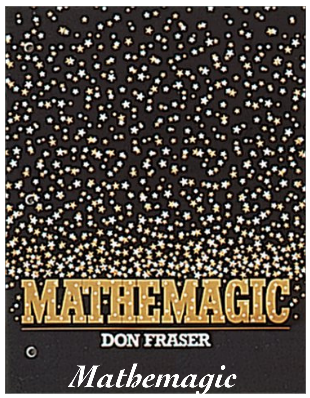
He fondly recalls two speaking engagements at NCTM conferences. One was in Boston, where he arrived late due to a snowstorm. He felt a bit scattered due to the weather. He got his overheads set up on a music stand, only to have them all fall to the floor in a mixed-up heap! He learned that day how to just "go with it," and to make the best of the situation in which you find yourself. In a second NCTM venue, he was in the audience, and the presenter did not show up. The chair asked if anyone had anything they could present to the packed room. Don opened his briefcase and shared a fun-filled hour of math and laughs.

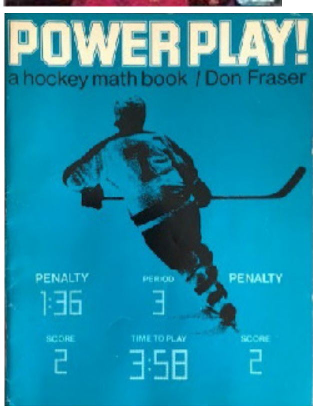
His publications include the following:

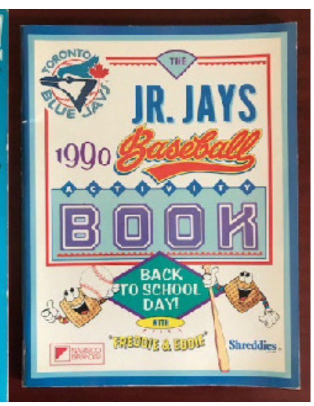




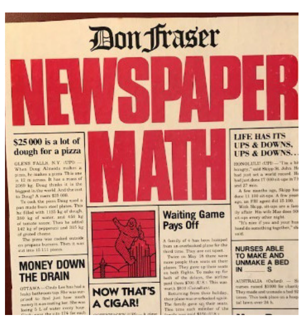








In 1990, 1991, and 1992, at a Blue Jay's home game, the Blue Jays gave the first 20 000 attendees a copy of Don's Blue Jays math books.



Don also wrote problem sets for *USA Today*, as well as for the *Toronto Star*'s "Math Language Activities," and he wrote blackline masters for "Newspaper Math."

For those new to OAME, or new to reading this column, it may seem like a lifetime of work that Don has shared with both OAME and NCTM. I asked him a few questions that may help others along their journey.

1. How did you become inspired to begin this work?
   He responded that five people from the early 70's helped to inspire him. They were Gary Hunt, Larry Ridge, Frank Ebos, Ken Weber, and Jack Le Sage.
2. What advice would you give those starting out to share their ideas?
   "Use humour and your own style to weave your presentation. Think about what the participants in your workshop need. Research the geographic area where you are speaking. Give examples that relate to them. As well, look for a catchy title such as "Math and Laugh," "Motivation, Meaning, and Mastery," or "Math Sense, Common Sense, and a Little Nonsense."

Don returned to the classroom twice in his math journey. The first time was in 1976/77 at Westview Secondary School in North York at Jane Street and Finch Avenue West. He feels that his returning to the classroom brought him joy, but humbled him to the plight of teachers. He used this knowledge to help guide his workshops. After retirement, Don worked for two professional development companies: Staff Development for Educators (SDE) in New Hampshire, and the Bureau of Education Research (BER) in Seattle.

A number of times, Don was selected by the students at the Faculty of Education, University of Toronto, as Teacher of the Year. Later, he became a Professor Emeritus at the University of Toronto.

This July 4, Don and Wendy will be celebrating their 60th wedding anniversary! As a parent, Don recalls causing each of his three children some anguish over the multiplication tables… noting that teaching is not easy!! He hopes his six grandchildren have had a much easier time with their math facts!

Don (who just turned 86 years young) signs each email he writes with "Don, Not Done!"

PS… Don would welcome any correspondence at dfraser@rogers.com or @DonFraser9.

NCTM offers virtual and in-person conferences.

- National Council of Teachers of English (NCTE) and NCTM Joint Conference on K–5/Elementary Literacy and Mathematics, June 17–19, 2024
- NCTM Annual Conference in Chicago, Illinois, September 25–28, 2024
- NCTM Regional Conference, Kansas City, Missouri, February 5–7, 2025

Until next time!

Yours in mathematical fun,
Jacqueline ▲

▲ Useful Theories to Know About: Achievement Goal Theory

JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Many of our students have personal goals. Their goals may be fairly broad, such as, "I want to be successful," or they may be quite specific, such as, "I want to get at least an 85% average to get into the program that I want at university." Students may not always verbalize their goals, and of course, their goals may change over the course of their high school journeys. Students' goals may be influenced by their backgrounds, peers, parents/guardians, teachers, and aspects of their current environments. A useful way to look at student goals is through achievement goal theory.

Typically, students will have short-term goals that lead toward their long-term goals. For example, a student whose long-term goal is engineering will have short-term goals related to doing well in calculus, which in turn will result in shorter-term goals, such as understanding derivatives, recognizing how to use calculus in optimization problems, getting good grades on tests in calculus, and so on. Achievement goal theory applies to both short- and long-term goals, and is a useful way for teachers to influence how students view learning.

Achievement Goal Theory

This theory identifies two main types of goals: mastery goals and performance goals. Mastery goals are intrinsic goals in which the student wants to master the material for the sake of their own learning. These goals indicate that a student wants to learn primarily because they want to really understand the content they are learning, and possibly apply it to new situations.

Performance goals are normative in nature. The student wants to look good compared to their peers, typically by getting higher marks than others. Performance goals can be further subdivided into performance-approach goals (wanting to do better than others) and performance-avoidance goals (not wanting to look bad in front of others). Students' goal orientation is often influenced by external forces, such as parental expectations, school reliance on marks as indicators of success; environmental fits, such as the matching of teaching styles with learner modalities; and other factors.

Why Should We (Math Teachers) Care about This Theory?

First, there is abundant evidence that students with mastery-goal orientations do better in school than those with performance goals. They more often become lifelong learners, are happier, and are more successful beyond school. Specifically for mathematics students, we know that students with positive attitudes are more engaged in their own learning and tend to achieve at higher levels. Students with mastery-goal orientations are more highly motivated to learn and understand mathematics content, which leads to greater achievement.

On the other hand, students with performance-goal orientations (either performance approach or performance avoidance) often achieve at lower levels, since they don't really focus on understanding material. We all know students who have achieved high marks, but didn't really understand material and couldn't apply it in non-routine situations. In extreme cases, students with performance-avoidance goal orientations can become disruptive or obstructive, just to avoid looking bad through their fear of poor performance.

A very important finding for teachers is that there is a lot of research that shows that a teacher's goal orientation can influence the goal orientations of their students. In a similar way to Carol Dweck's growth-mindset theories, if a teacher demonstrates through their words and actions that a mastery-goal orientation is what is valued in class (learning for understanding), students will frequently adopt the same mastery-goal orientation as their teacher. Alternatively, if the teacher demonstrates a performance-goal orientation, for example, by emphasizing marks over understanding, students will tend to do the same. This relationship is so strong, it is sometimes called the "class-goal orientation."

Students frequently will not verbalize their achievement-goal orientations. However, if a teacher clearly indicates a mastery-goal orientation to learning, their students will often become more successful than they otherwise would be.

Continued on page 19

▲ Events at the Fields: Undergraduate Education, Framing Mathematics, and Research Day 2024

PAMELA BRITTAIN
pbrittai@fields.utoronto.ca

Pamela Brittain, Ph.D., is an educator, author, illustrator, presenter, and math lover, who currently works for the Fields Institute. She has been involved in STEM education for over 20 years and strives to make math accessible and enjoyable for everyone.

After a month-long break for the December holidays, the Fields MathEd Forum was back in an inspiring form for January. Our first session back, however, started with the sad news of the passing of Dr. Pat Rogers, former Dean of Education at the University of Windsor and Associate Vice-President, Teaching and Learning at Wilfred Laurier University. Some of Pat's former colleagues provided a moving memorial and listed some of her accomplishments and contributions to the world of teacher education.

January: Undergraduate Mathematics

The theme of the January sessions was undergraduate learning and how to make it more engaging and accessible to students. The first talk, "Supporting the Transition between Mathematics Courses and Physics Courses in the First Year of University" by Pauline Hellio, Ghislaine Gueudet, and Aude Caussarieu (Université Paris-Saclay, France), focused on the differences between how mathematics is taught in a math course and how it is applied in a physics course, and the issues this can cause for students. The main difficulty they found is transferring theoretical mathematics knowledge into practical applications in a physics class. They then discussed methods for easing this transition and different ways to teach undergraduate math.

The second talk, "'I Am Learning So Much from This Peer Collaboration' – But Do You Though?" by Igor Kontorovich (University of Auckland, New Zealand), was about collaborative learning and its effectiveness in mathematics courses. Using automatic software (STACK), the researcher looked at how interactions between peers in a collaborative math lesson played out. One of the interactions presented was between a confident math student working with a non-confident student and how leader/follower roles quickly developed. At the beginning, the collaboration looks to be successful, as both students are respectful of each other, and are working well together. This causes issues, though, when they encounter a problem with which the more confident student isn't as confident. When that student attempts to pass it on to the other student, it quickly gets passed back, and both start to become non-confident and confrontational toward each other. The presenter spoke of how collaborative learning is more complicated for students than is often assumed, and that there are many considerations that need to be taken into account.

The morning session ended with a panel discussion, where the speakers discussed "Perspectives on Teaching Mathematics Courses." It included instructors of a large (2000- to 3000-student) calculus course and instructors of smaller, more targeted topic, math courses. Some of the topics and themes discussed included asking, "What does it mean to be authentic in mathematics?" and "What does numeracy mean in mathematics?" It was a lively debate with some differing opinions, but overall, it created some interesting "food for thought" when considering what it means to teach mathematics.

The day ended with another panel discussion, this time on the topic of "Curriculum and Assessment – Conundrum and Envisioned Future," and there was lots of discussion on how Artificial Intelligence has changed assessment and evaluation, and what it means to "cheat" now (and should we modify what is considered cheating, which caused some debate between the panelists). There were also discussions about the recent PISA and OECD results and what they mean, including what mathematics content is being focused on with these tests and what is being focused on in the classroom. The role of social responsibility in mathematics also came up and sparked some insightful debates and discussions. One panelist asked the question, "Who is sitting in on math classes and why are they there?," and how we tailor math classes to meet these needs. There was also a discussion about an "anarchist framework" in teaching math, and how this applies to things such as grading.

Overall, the January session contained a lot of interesting points to think about and, while the focus was on undergraduate courses, many of the topics discussed applied to secondary and elementary math classes as well.

February: In and of the World—Framing Mathematics

The February session started with a presentation by Eva Thanheiser (Portland State University, Oregon) titled "What Do We Mean By 'Mathematics'?," where she started with a discussion on population distribution and some of the biases inherent in what we include (and don't include) when we talk about mathematics. She showed the audience an image, a quotation, and some statistics, and asked if they were considered mathematics, to which the audience was split on their interpretations. Her talk also introduced three frameworks that she used throughout the talk to highlight many ideas.

Her talk discussed issues related to mathematics and social context, including proportionality in prison populations, and even in comparing advanced and remedial mathematics classes. She also made note of the fact that, even when discussing social issues in mathematics, she still gets asked the question, “Is 2 + 2 still equal to 4?,” showing a focus on mathematics as numeracy and arithmetic by many, including the many schisms that exist between researchers and practitioners about where we should focus in math, and even how math is defined in the curriculum.

Her talk then asked the audience to participate in a “thought experiment” looking at the percentages and ratio of the population of the world to the population of Jewish individuals, both historically and now, and to question if the Holocaust affected these ratios. The talk then ended on exploring the idea of student-centred learning in the context of the three frameworks and asked the audience to consider where the focus should be, and how to implement some of these ideas.

The next talk, “What’s Mathematics Education Got to Do with the Ecosystem Crisis?” by Richard Barwell (University of Ottawa), used the culling of wolves in the United States to generate a discussion on how mathematics played a role in justifying the culling. He then asked the audience to think about how mathematical modelling affected the wolves, how the wolves affected the math, how math was used, and what impacts it had. He then shared some research theories that structured the conversation around the question, “The mathematics education we need?,” and asked the audience to consider math’s role in consciousness and responsibility. The presentation ended with some thought-provoking questions on things like “What do we teach when we teach math?,” which really left the audience pondering mathematics and our relationship to it.

The day ended with some small-group discussions on the topics raised throughout the day.

March: Research Day

Finally, the March session was all about research and included a wonderful keynote speech on “The Advancement of Research on Teaching Mathematics” by Agida Manizade (Radford University, Virginia). Her talk was based on her book, *The Evolution of Research on Teaching Mathematics* (https://link.springer.com/book/10.1007/978-3-031-31193-2) and took us through a number of different, common types of units of analysis used in mathematics research within cultural, epistemological, and digital contexts. She also spoke about the role of understanding what variables influence the outcomes in research and recognizing which ones fell within and outside of teacher control. She also discussed how cultural context affects the outcomes and how we need to recognize the lens through which we are viewing the research.

The rest of the day was spent in individual presentations, where many math researchers from across Canada, and the world, gave talks, short oral presentations, and poster sessions showcasing the work they’ve been doing. Unfortunately, only the keynote was recorded, but you can read short abstracts for the sessions at www.fields.utoronto.ca/activities/23-24/meforum-Mar.

Last Session in April

As of the time of writing this article, we are left with only one more MathEd Forum to come in April. The MathEd Forum will then take a hiatus until September, and when we return, we’ll have the Margaret Sinclair Award Presentation by this year’s winner, Dr. Lauren DeDieu. You can read more about her and the award at www.fields.utoronto.ca/news/Dr-Lauren-DeDieu-receives-2024-Margaret-Sinclair-Memorial-Award.

We encourage you to come and join the conversation. The Fields MathEd Forum convenes on the last Saturday each month from January to April, and September to November, and are held both in person at the Fields Institute (222 College St., Toronto) and online. Some sessions, like the Research Day, are held fully online.

You can register to attend, learn about the theme, read presenter abstracts, and also view select recordings of past sessions online at www.fields.utoronto.ca/activities/23-24/meforum.

100-Year ICM Celebration

As mentioned in the editorial, this year marks the 100th anniversary of the ICM being held in Toronto, brought here by Charles Fields himself (our namesake); so how could we not celebrate?! Join us in August for “Forward from the Fields Medal,” where we will be celebrating everything math and bringing back Fields Medalists from across the decades. In addition to the Scientific Symposium, the event will also contain cultural events such as concerts, film screenings, and even a boat cruise. **Please note:** There is a cost to attend portions of this event (www.fields.utoronto.ca/ activities/24-25/FFFM-2024). You can also learn more about John Charles Fields and his role in the 1924 ICM in this short video: www.fields.utoronto.ca/news/Who-Was-John-Charles-Fields. ▲

▲ Meaning Making in the Classroom: What Does Understanding the Problem Really Entail?

DAVID COSTELLO
david@costellomath.com

David is a principal, who taught and had various board roles in Prince Edward Island. David has also instructed university courses regarding curriculum, differentiation, mathematics, and literacy. He facilitates professional learning in mathematics and school development. Dr. David Costello recently published four books: Messing Around with Math: Ready-to-Use Problems That Engage Students in a Better Understanding of Key Math Concepts, Mathematizing Student Thinking: Connecting Problem Solving to Everyday Life and Building Capable and Confident Math Learners, *and* Making Math Stick: Classroom Strategies That Support the Long-Term Retention of Math Concepts.

I have spoken with quite a few math educators this school year. Regardless of the grade level they instruct or the number of years they have been in education, many of these educators highlight the importance of having students understand the problem they are assigned. Without this understanding of the problem, students may have little to rely upon when working toward a solution.

It is through understanding the problem that students can navigate the problem-solving process. Once the problem is understood, students can identify what is being asked of them. Students can then select a plan of action that will lead them to the preferred outcome of the problem. If students do not understand the problem, how can they self-monitor the validity of their work? And, without understanding the problem, how can students determine their work is accurate? Understanding the problem is a cornerstone of problem solving.

Consider the importance of "understanding the problem" by thinking about the amount of time educators put into this part of problem solving. Are we providing students with opportunities to dig into the problem and to strengthen their understanding of the problem? Are we providing students with opportunities to share their understanding of the problem? Or, is this part of the problem-solving process rushed through, and more attention is given to which plan of action students have selected to solve the problem?

Consider understanding the problem from the perspective of the student. Although they may be told to read the problem carefully and to read the problem more than once, is this really being done? Or, are students quickly skimming through the problem, looking for highlights, so that they can quickly get to *doing the work* of selecting a plan action and then enacting it? I am told by many students that they don't have to read the problem more than one time to get to what it means. They can just move forward instead of "wasting time."

Regardless of the problem type students are assigned, whether it is defined or ill-defined, understanding the problem is crucial to making meaning in the classroom. As such, what follows is a brief overview of understanding the problem, and concrete strategies that can be applied to your instructional toolbox, so that students can strengthen their understanding of the problem and use this to become confident and capable math learners.

Understanding the Problem

Consider the time spent in literacy classes when students are reading a text. It is common, at all grade levels, that once a text is read, either by oneself or by someone else, students are asked to share their understanding of the text. Teachers do this to determine whether students have an accurate grasp of what was being presented within the text. If students do not seem to understand the text, the teacher will use this time as an opportunity to support students in growing their understanding of the text.

This is what we need to do in math. We need to ensure students understand a problem, before they move onto selecting a plan of action. When students are presented with a problem, regardless of the problem structure, they should be able to clearly articulate what the problem is about and what is being asked. A goal for students is the ability to restate the problem and determine whether there is enough information provided in the problem to make an informed decision as to a plan of action (Costello, 2022).

I have identified four strategies that can be used to support students in understanding the problem: questioning, paraphrasing, finding the match, and think-aloud. These strategies are not exclusive and can be used in conjunction with one another.

Questioning

The use of questioning can aid students in understanding the problem. Questions can be self-directed or asked by others. What is imperative is that students consider questions and how they relate to the problem they encountered. Consider the following questions that could be used to support students in understanding the problem

(Butler Wolf, 2015):

- Have you read the whole problem?
- What is the problem about?
- What are you supposed to find out?
- Can you rephrase the problem in your own words?
- Can you draw a picture that represents the problem?
- Can you represent the problem using concrete objects?
- Have you seen a problem similar to this one?
- Does this problem remind you of anything you have encountered?

The use of questioning to support students in understanding the problem does not have to be extensive. The point in this strategy is to encourage students to take time to ensure they understand the problem that they have been assigned.

Finding the Match

It is important to identify the learning goal of the lesson. Sometimes when I am in a class, the learning goal is not to successfully solve a problem, but to strengthen a particular aspect of problem solving. For this column, let's consider understanding the problem as the aspect I want students to strengthen.

Finding the Match is an instructional strategy I have used at many different grade levels. As the teacher, I will post a problem on the whiteboard. I will then post three different paraphrases of the problem, with only one of the three being an accurate paraphrasing. Students are then asked to read the original problem and then read the three different paraphrases. Students will identify which of the three paraphrases most accurately align with the original problem. Students must be ready to rationalize their response.

Consider the following example as an illustration of Finding the Match.

Original Problem

Elm Grove Primary is a school with a population slightly over 200 students. To raise money for a field trip, the school decided to sell calendars. For each calendar sold, the school makes a small profit. If most of the students sell a calendar, about how much money would the Elm Grove Primary raise?

Option 1

Elm Grove Primary's 200 students sold calendars for a school fundraiser. Most students sold a calendar. How much money was raised by the school if a profit was made on each sale?

Option 2

Most of the Elm Grove students sold a calendar for a school fundraiser. The school made a little bit of profit for each calendar a student sold. If there are a little over 200 students in school, how much money was made through the calendar fundraiser?

Option 3

Calendars were sold as a fundraiser for Elm Grove Primary. If most students sold a calendar, how much money did the school raise?

If you provided this example to your class, which of the three paraphrasing options is most appropriate for the original problem. What rationale would you expect students to provide?

Paraphrasing

While paraphrasing is typically a strategy considered when students read, there are opportunities to apply paraphrasing in math. When thinking about math, students can be asked to paraphrase a given problem in their own words. This may not be as simple as you imagine. To paraphrase a problem, students must understand the parameters of what was presented in the problem and be able to rephrase it in language they use. Paraphrasing is the ability to consider what was presented, think about it in a way that makes meaning, and then reword it using different words, while not losing the intent of the original problem.

By paraphrasing, students must take time to reflect on what was presented in the problem and to articulate it in their own words. Being able to paraphrase a problem is a clear indication that students have read the problem and have understood what is being asked of them. The ability to paraphrase a given problem should not be considered as an add-on, but instead as a crucial component of understanding the problem. If a student is unable to paraphrase a problem, one must really question if the student understands the problem. And, if the student doesn't understand the problem, why are we having students move onto crafting a plan of action in the problem-solving process?

An example showing how paraphrasing can be included in the problem-solving process can be seen in the examples that follow (Costello, 2022).

The graphic organizer at the top right of page 15 is representative of Pólya's four stages of problem solving. The graphic organizer at the bottom of page 15 accommodates paraphrasing to support understanding of the problem. I have seen this graphic organizer being used in several different ways. First, before students move on to understanding the problem and crafting a plan, they must share their paraphrasing of the problem with either another student or teacher. Second, the teacher can ask students to use their paraphrasing as a compass to self-monitor their thinking, while they work through the problem.

Think-Aloud

Think-aloud is a strategy that can be used to make the implicit explicit (Costello, 2019). Used by either the teacher or students, think-alouds provide opportunities to highlight the critical decision points that occur when engaged with a problem. Used by the teacher, think-alouds enable the teacher to highlight aspects of their thinking so that students can recognize the different thinking points within the work. Used by the student, think-alouds provide an opportunity for students to orally share what they are thinking. In short, think-alouds support metacognition.

When students engage in a think-aloud, they must meticulously go through the many decisions related to the task. For our instructional goal of supporting student understanding of the problem, students would read through the problem and think through the various aspects of the problem. This allows students opportunities to re-engage with the problem and to clarify their understanding of it.

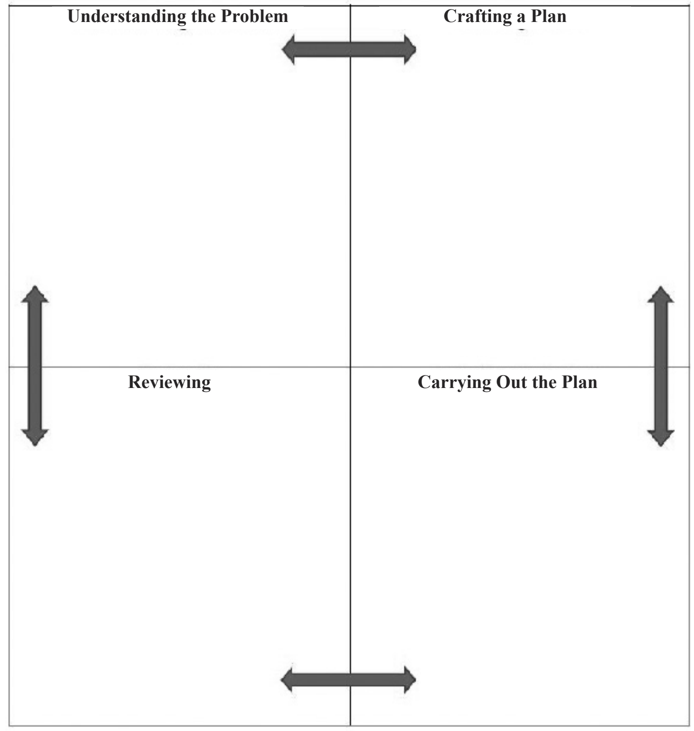
When engaging students in a think-aloud, be mindful that the problem provides opportunities for stumbling blocks. Think-alouds are most beneficial when students must critically reflect on their thinking and then share it with others. Perhaps offer students time to share a think-aloud, either with you or another student, as it pertains to their understanding of a problem.

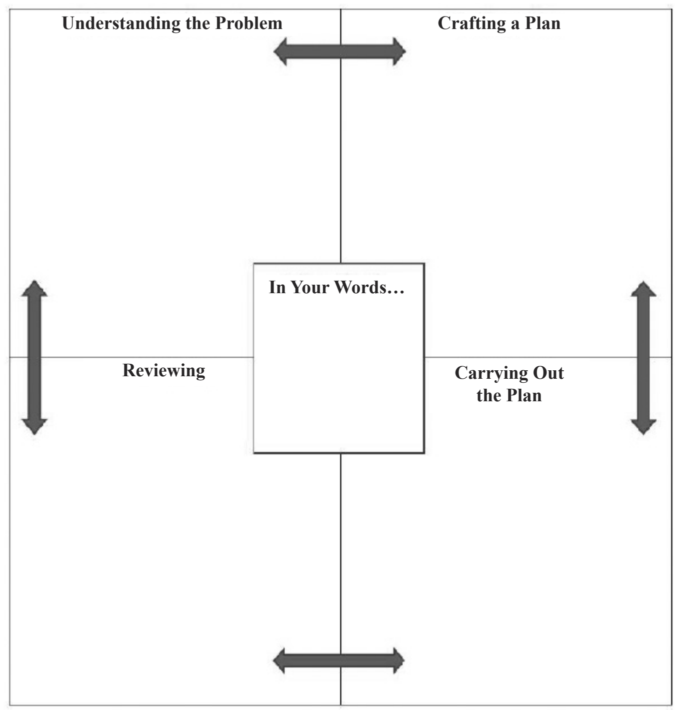
Final Thoughts

As can be seen within this column, understanding the problem is not a part of the problem-solving process to rush through. Students must have an accurate understanding of the problem to work through it toward a solution. If students do not accurately understand the problem, how can we expect them to make informed decisions through the problem-solving process. If the foundation of problem solving is understanding the problem, students must take the necessary time to make sense of it.

Too often we rush through understanding the problem when problem-solving. Take time. Try one or more of the strategies I have shared in this column. I promise you that the time taken is a valuable investment, one that will have significant value added—students will use their understanding of the problem to guide their thinking as they work through a problem.

I look forward to continuing the discussion of mathematical learning and instruction via my website (www.costellomath.com), email (david@costellomath.com), or X (@dr_costello).





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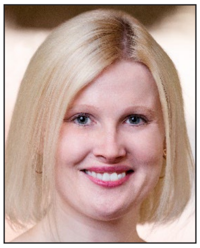
▲ Mb4T (Mathematics by and for Teachers): Flexible Thinking About Subtraction



JENNIFER HOLM
jholm@wlu.ca

Jennifer Holm is an Associate Professor at Wilfrid Laurier University and works with Primary/Junior and Junior/Intermediate pre-service teachers, as well as in the field supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching with both pre-service and in-service teachers. She focuses on the beliefs and opinions that pre-service teachers hold about mathematics and teaching and the connection they have to past experiences. She uses this research to support future teachers in developing beliefs and knowledge that will encourage and support effective teaching practices in mathematics.

The focus of this column is different meanings of subtraction and how expanding the definition of subtraction from only "take away" leads to more possibilities for flexible thinking related to the operation. As a former Kindergarten teacher, I know how simple it is to equate subtraction to "take away" when introducing the subject. I remember teaching the unit, using fireflies, with small fireflies I cut out to use for our mathematics. I could set up word problems such as, "I had ten fireflies in a jar. When I opened the lid to check on them, three flew away. How many fireflies were left in the jar?" This made a scenario for students to take ten of the cut-out fireflies and then take three of them, that flew away, from the total, leaving seven in the jar. This definition is helpful for beginning to understand subtraction because it is intuitive for children to "remove" something from the total amount to find what is left. In this column, I will explore how expanding this definition can help in later years with more complicated subtraction problems.

Most of you have likely seen the memes, or had it mentioned in professional development, about students who use the standard procedure for subtraction to solve 1001 – 997 (Figure 1), instead of thinking about the numbers. Oftentimes, it is noted that inflexible understanding of subtraction, and reliance on the standard procedure impedes the ability to think and reason when faced with different questions. The standard procedure is helpful to learn, and I have used it myself many times as an adult, but what I am saying is that sometimes it is not necessary or helpful. As an adult, when I learned about teaching mathematics, I learned more about different subtraction definitions and found that many times I could just think about the numbers to determine an answer, without using the standard procedure or a calculator. Without different ways of thinking about subtraction, though, I would not have had the opportunity to consider alternatives to the procedure I had memorized in school.

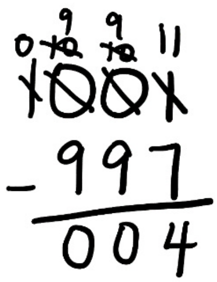


Figure 1: *Using the standard procedure to subtract: 1001 – 997*

There are two other meanings of subtraction: "difference" and "whole and part." To illustrate these definitions, I will provide contextual problems. I suggest doing something similar in your classroom so that you can help students see and develop the different definitions. It is not important for children to know the names of the definitions, but rather, to understand them. This understanding can help build a more flexible understanding of number and the subtraction operation itself.

Consider the following three contextual problems, each showing a different definition of subtraction:

1. There are fourteen cookies on a plate. A child eats five of the cookies. How many cookies are left?
2. Kayla is fourteen years old and Isobel is five years old. How many years apart are the two girls?
3. There are fourteen treats left on the bake-sale table. Five of them are cookies. If the rest are brownies, how many brownies are left?

All three problems have the same numerical solution or answer, but think about how you would model the process for solving each of the problems if you were a child. You are probably noticing that each one is slightly different. There is an example of how to model each problem in Figure 2.

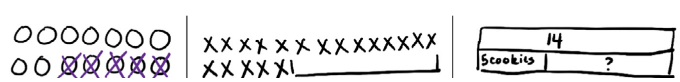


Figure 2: *Sample models for each of the contextual subtraction problems*

In Figure 2, problem number 1 is portrayed as the typical "take-away" type of problem. In a model, a child could cross out or remove the five cookies that were eaten to show what remains. Problem 2 looks at the "difference" between two numbers. This one relies on a different definition when you are comparing the numbers to see what is between them. A child may solve this one by counting up or down between the numbers to see how they differ. The third problem looks at whole–part definition. This is sometimes referred to as a "bar model" for subtraction. In this case, you know the total amount and one part, and you need to determine the other part. Again, it is the different contexts that help to push the different models and definitions. Simply giving the subtraction question of 14 – 5 pushes toward a memorization of facts only. Knowing the simple facts, and being able to automatically answer them, is an important part of learning mathematics—and there are a number of enjoyable ways to help students learn them. But the purpose of this column is to think about what it means to subtract, so that you have different ways to find solutions to subtraction problems, since sometimes one approach may be more efficient than other approaches.

Knowing that subtraction is used to find the difference between two numbers creates a lot of possibilities for thinking about numbers flexibly, especially using the open number line. For the rest of this column, we will look at larger numbers, using an open number line to subtract, and consider different ways to determine answers to the problem. Although the open number is extremely helpful to organize thinking when first subtracting, the goal would be for students to start doing the mathematics in their heads, so that they can determine the answers without a full reliance on the standard procedure or pencil and paper for solving a problem.

Before looking at a subtraction problem, there are a few points to keep in mind about the open number line to help with organizing thinking with numbers:

1. The open number line is just a segment of an entire number line, so it continues in both directions from the interval being used.
2. The open number line is not like a ruler or other number line, where there are demarcations for numbers or increments on the number line. It will only have the starting value where you begin your thinking, and then label the parts as you move through the questions. If you look at the different colours in the figures that follow, as well as the notes in the figure captions, you will see the different steps in the thinking.
3. It is helpful to put the question that you are working with on the page to help with reminders of where you are going with the problem, since you do not label all the parts used in the thinking.
4. There are different ways that an open number line can be used, and it is important to keep the spacing and "hops" relative in size (e.g., a hop of 2 would be smaller than a hop of 20). The accuracy is not essential, but it is good for visualizing the numbers and spacing to be cognizant of the size differences with numbers.

To illustrate the point about subtraction, consider the subtraction problem of 252 – 78. We could always use the standard procedure to solve the question, but it is not the only way. Take a moment now to think of other ways you might find the answer in your head.

One way is to count up to 100 first (78 + 2 = 80, then another 20 gets to 100), which means that 78 needs 22 more to make 100. Then, from 100, you need to add 152 to make 252. The solution is 22 + 152 = 174.

Figure 3 shows how an open number line can be used to keep track of this question. The open number line starts by labelling the 78 and the ending point of 252 in an attempt to keep on track for solving the question.

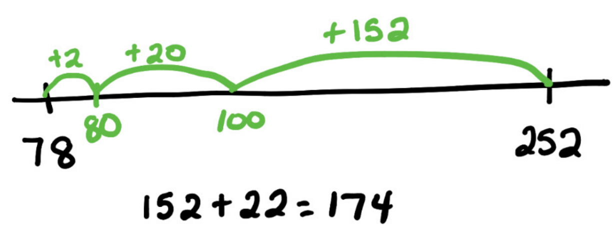


Figure 3: *One way to calculate 252 – 78 by adding up. The open number line begins with 78 and 252 placed on the line, and then shows adding 2 to get 80, 20 more to get 100, and then 152 to reach the end. This shows that 252 – 78 = 2 + 20 + 152 = 174.*

A second way is to consider changing the problem, but keeping the distance between the two values the same. If I remove 52 from both numbers, the problem is now 200 – 26, which is a simpler subtraction problem. 200 – 20 is 180, and then subtracting another 6 (200 – 26 = 200 – 20 – 6 = 180 – 6) would be 174. Figure 4 shows this on an open number line by starting with the original problem of 252 – 78 and showing the change, and then a second open number line to show the subtraction.

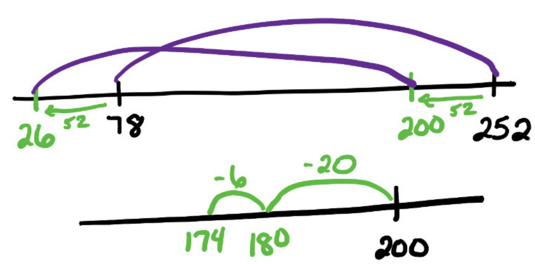


Figure 4: *Another way to calculate 252 – 78, using an open number line to keep track. The first open number line shows 252 and 78, and then how removing 52 from each number keeps the initial distance the same: 252 – 78 = 200 – 26. The second number line starts with 200 (the new value), and then subtracts 20 and then 6 to show that 200 – 26 = 174.*

A third way is to subtract in parts. For example, start by subtracting 50 from 252, to make 202. Then subtracting another 20 would be 182. Then I could subtract 2 to make 180, and then the final 6 to make 174. Figure 5 summarizes this way of considering the question with an open number line by starting with 252 on the number line and then subtracting the 78 to determine an end point.

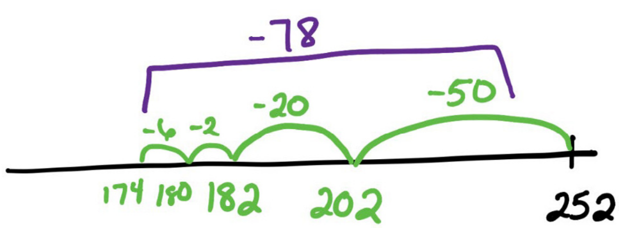


Figure 5: *Using an open number line to subtract 252 – 78 in parts. 252 is initially placed on the open number line, and then the number 78 is broken up to subtract in parts, to determine the ending point.*

A fourth way could be to subtract 100 first (since 78 is close to 100), making 152, and then add back the extra that we removed. We could then add back 2 to make 154, and then the additional 20 to make 174 (we subtracted 22 too many initially, since we should only have subtracted 78). Figure 6 shows how this might look on the open number line. Initially only 252 is placed on the open number line, and then the different steps show the subtraction.

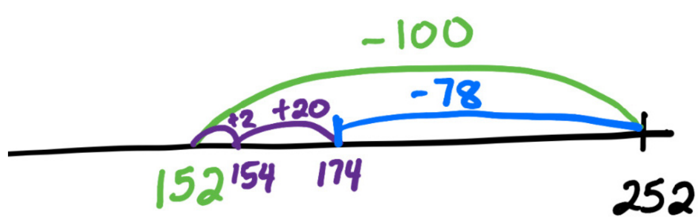


Figure 6: *Using an open number line to show subtracting 252 – 78. 252 is placed on the open number line, and then 100 is removed to get 152. The next step is to add back the extra parts that are subtracted. To illustrate that 78 was subtracted, a different shade was used to show the -78 (which is equal to 100 – 22).*

Any of these methods could break down the question in different ways. For example, using the second method, we could have just added 2 to each number to make the question 254 – 80, which, for some, would also be a simpler way to subtract. These examples were just to consider how we could use different ways to find the answer.

I am not claiming that any of these examples are more efficient than the standard procedure for 252 – 78. This was merely an illustration to compare different ways to consider the same problem. The idea would be to consider if there is a more efficient way for a particular problem. For example, if you consider the first method and the problem 423 – 375, it is more efficient to realize that adding 25 to 375 results in 400 and then adding another 23 results in 423, so the solution is 25 + 23, or 48 (423 – 375 = 48). The idea is not to practise each method or way with each problem, but once students know some different ways to think about subtraction, have them consider what might be an effective method to solve. Again, the goal is flexibility in their thinking, not to practise different methods over and over again.

Return to the standard procedure shown in Figure 1 and consider 1001 – 997. We could use an open number line to solve, but the goal is that after multiple different problems and all the work with the open number line, it would not be necessary. Thinking flexibly, a student may consider that 1000 – 997 is a difference of 3, so 1001 – 997 is one more, so the answer is 4. Students may also count up from 997 to 1001 and find the answer is 4 (perhaps by first counting to 1000 and then 1 more). Having a stronger understanding of subtraction, beyond only memorized facts and procedures, helps in allowing students to first think about the numbers and then come to an answer quickly.

In conclusion, expanding the definition of subtraction beyond just thinking about subtraction as “take away” gives students a different perspective for solving subtraction problems, and allows for flexible thinking. Having the open number line to scaffold their thought process as they first start working with different problems will help give a mental picture that eventually can be removed as students are able to reason with the numbers.▲

Continued from page 10

If You Want to Learn More

The relationship of teachers’ goal orientations with the goal orientations of their classes:

Achievement goals related to other student affects:

Dull, R.B., Schleifer, L.L.F., & McMillan, J.J. (2015). Achievement goal theory: The relationship of Accounting students’ goal orientations with self-efficacy, anxiety, and achievement. *Accounting Education, 24*(2), 152–174. doi.org/10.1080/09639284.2015.1036892

Shim, S.S., Cho, Y.J., & Cassady, J.C. (2013). Goal structures: The role of teachers’ achievement goals and theories of intelligence. *The Journal of Experimental Education, 81*(1), 84–104. doi.org/10/1080/00220973.2011.635168

The relationships of achievement goal theory to other student affective dimensions and achievement:

Wolters, C.A. (2004). Advancing achievement goal theory: Using goal structures and goal orientations to predict students’ motivation, cognition, and achievement. *Journal of Educational Psychology, 96*(2), 236–250. doi.org/10.1037/0022-0663.96.2.236 ▲

▲ What’s the Problem? Playing with Processes: Part 4

SHAWN GODIN
GodinMathStuff@gmail.com

Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. In this issue, we wrap up our four-part series focusing on the mathematical processes. In the last three issues, we highlighted the processes **Reflecting** (Godin, 2023a), **Representing** (Godin, 2023b), and **Connecting** (Godin, 2024). In this issue, we will tackle **Reasoning and Proving**.

Last issue was the first time since this column started back in 2006 that no homework was assigned. I had encouraged readers to send me some of their favourite problems that deal with **Reasoning and Proving**. As I start writing this column, my last column has just been made available online, but I haven’t received my copy yet. As such, I have not received any problems from the readers in time for this column. Do not despair! If you send something to me and I like it, I will either mention it or feature it in a future column (more on that at the end of the column). In this issue, we will look at various examples of **Reasoning and Proving**, using some problems I have previously investigated in this column.

As in the three last issues, the reader should keep in mind George Pólya’s four steps in problem solving from his book *How to Solve It* (Pólya, 1957). Here they are provided as a reference, with some of his guiding questions reproduced below in italics. As in the last issue, relevant mathematical processes are noted in bold within parentheses throughout the column. The keen-eyed reader will notice that the wording in this issue and the last is slightly different from the first two columns in the series. *Gazette* Editor, Tim Sibbald, suggested the changes last issue. As he pointed out, they still embody the spirit of Pólya, while encouraging students to actively think about the problem-solving process(es).

1. Understand the problem. It is impossible to solve a problem if you don't understand what is being asked, the notation, or terms in the problem. This involves, in some cases, accessing prior knowledge. (**Reflecting**)
2. Devise a plan. (**Selecting Tools and Computational Strategies, Representing, Reasoning and Proving**) *When have you seen it before? Think about a related problem.* (**Reflecting, Connecting**)
3. Carry out the plan. As you work through each step of your plan: *Describe how you know that step is correct.* (**Reflecting, Communicating**) *How can you prove that it is correct?* (**Reasoning and Proving**)
4. Looking back. *How can you check the result?* (**Reflecting**) *Can you show this result differently?* (**Reasoning and Proving, Representing**) *How could you use the result, or method, for some other problem?* (**Connecting**)

As the format of this column is a bit different, we will not be following these steps as closely as in the past three columns, but we will reference them from time to time.

Another small change readers may have noticed in the March issue is that the EduGAINS website is no longer available. As a result, I have made the material that I downloaded from the EduGAINS website available on my Google Drive at www.drive.google.com/drive/folders/1gY76NCxf5LxyG5Ugy-gEY_NxCovLtQET?usp=drive_link.

We will start by examining the rubric for **Reasoning and Proving** from the Ministry of Education (Ontario Ministry of Education, 2008), shown in Table 1.

There is a LOT going on here. This should not be surprising, as *reasoning* is a synonym for *thinking*. As we have mentioned throughout the series, the processes are **not** discrete entities. There is a multitude of overlaps between all the processes (I wouldn't dare try to draw a Venn diagram showing the relationship between the processes). The rubric allows us to focus on some of the major places where we will see Reasoning and Proving in action:

1. *Forming hypotheses and conjectures.* This can happen in many places during the problem-solving process. It can happen during an investigation while gathering and examining data. In this case, the hypothesis becomes the problem that we are trying to solve (Pólya's step 1 and pre-step 1).

   It can happen when we are presented a problem to solve, and our hypothesis is how various mathematical concepts will come together to yield our result. In this case, the hypothesis becomes part of the planning process (Pólya's step 2).
   It may happen during the process of solving a problem. One of the steps that leads to the answer may suggest a pattern of its own that can be explored. In this case, we might be generalizing the pattern (Pólya's steps 3 and 4 with **Reflecting**).
2. *Interpreting/expressing.* This occurs throughout the problem-solving process, anytime we are faced with some mathematical notation, language, or graphics that we have to understand in the context of the problem. Interpreting primarily occurs at the start (Pólya's step 1), whereas expressing would be nearer

Table 1: *Rubric for Reasoning and Proving*

Reasoning and Proving				
Criteria	Level 1	Level 2	Level 3	Level 4
Formulates and defends a hypothesis or conjecture	Forms a hypothesis or conjecture that connects few aspects of the problem	Forms a hypothesis or conjecture that connects some of the pertinent aspects of the problem	Forms a hypothesis or conjecture that connects pertinent aspects of the problem	Forms a hypothesis or conjecture that connects aspects of the problem with a broader view of the problem
Makes inferences, draws conclusions, and gives justifications	Makes limited connections to the problem-solving process and models presented when justifying answers	Makes some connections to the problem-solving process and models presented when justifying answers	Makes direct connections to the problem-solving process and models presented when justifying answers	Makes direct and insightful connections to the problem-solving process and models presented when justifying answers
Interprets mathematical language, charts, and graphs	Misinterprets a critical element of the information, but makes some reasonable statements	Misinterprets part of the information, but makes some reasonable statements	Interprets the information correctly and makes reasonable statements	Interprets the information correctly, and makes insightful statements

the end when you are making it easy for others to interpret your solution (**Communicating, Representing**). Although the problem-solving process seems focused on “answering” some question, it is important to recall that an answer without support isn’t worth much. The solver should be able to convince themselves that their solution is correct. The validity of the solution is strengthened if the solver can convince others of its accuracy.

3. *Making inferences and drawing conclusions.* This arises when we bring together some ideas or observations and deduce a new result from them (**Connecting**). This usually occurs when we are in the thick of things, carrying out our plan (Pólya’s step 3).
4. *Justification.* This is the process of supporting or reinforcing your statements and thoughts. We are doing this when we defend a hypothesis or conclusion. This can be done by tying things back to known facts or by defending your ideas logically. Justifying is the crux of the **Reasoning and Proving** process (**Connecting, Communicating**).

We will see **Reasoning and Proving** in action by re-examining a couple of problems from past columns. We will start with the following problem (Godin, 2011):

Ancient Egyptians wrote all fractions in terms of distinct unit fractions (that is, in terms of distinct fractions with numerators of 1). For example, instead of writing $\frac{11}{12}$*, they would write* $\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$*. The unit fraction* $\frac{1}{2}$ *can be written in terms of other unit fractions as* $\frac{11}{12}$*. Find some other unit fractions that can be written as the sum of two unit fractions.*

Some students are very good at finding patterns. Through playing and finding some of their own examples, or just by looking at the example $\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$, students may come up with the following conjecture: *Any unit fraction is equal to the sum of the unit fraction whose denominator is one more than the denominator of the original fraction and another unit fraction whose denominator is the product of the denominator of the original fraction and a number one more than it* (**Reasoning and Proving**). That is quite a mouthful! I doubt any real student would ever say that (if one of yours does, please let me know). However, they may say something like: “Notice in the example $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$, that 3 = 2 + 1 and 6 = 2 × 3. This happens all the time. So, if I pick a number like 5, then 5 + 1 = 6 and 5 × 6 = 30, so $\frac{1}{5}=\frac{1}{6}+\frac{1}{30}$.”

Other than wild, random guesses, students’ conjectures will have a degree of reasoning behind them. However, like many of the mathematical processes in action, as the reasoning is going on in their head, it is hard to judge the depth of it unless the student clearly communicates it to you (**Communicating**). As this is the beginning of the problem-solving process, students may not think of including their reasoning behind their conjecture in their solution. Teachers can get at this thinking by direct questioning during or after the problem-solving process. Another way to access their thinking is to have students set up a problem-solving journal. In the journal, they can present their solutions to problems, with annotations. Students can be prompted to share their thinking by considering what a teacher would ask about their solution, such as: “Why did you do that?”; “Have you considered…?”; “Is there another way…?” These journals can take many forms, such as the traditional booklet or even through video, with the students doing the annotations verbally.

It is one thing to come up with a hypothesis, but it is another thing to defend or *justify* it (i.e., prove it). It is natural for students to use examples as justification, or “proof,” of their conjectures. For the current conjecture, many students may give something like

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{12}$$
$$\frac{1}{4}=\frac{1}{5}+\frac{1}{20}$$
$$\frac{1}{5}=\frac{1}{6}+\frac{1}{30}$$

as their justification. While the identification of more cases should increase our confidence that our conjecture is true, it is not a *proof* that the conjecture holds for all possible cases. A student comfortable with algebra (**Representing**) can complete this proof by letting *n* represent a natural number that is the denominator of the fraction on the left-hand side of the equation. Then, if we work with the right-hand side, we get:

$$\frac{1}{n+1}+\frac{1}{n(n+)1} = \frac{n}{n(n+)1}+\frac{1}{n(n+)1}$$
$$\frac{1}{n+1}+\frac{1}{n(n+)1} = \frac{n+1}{n(n+)1}$$
$$\frac{1}{n+1}+\frac{1}{n(n+)1} = \frac{1}{n},$$

which shows that the conjecture is true, no matter the value of *n* (**Reasoning and Proving**). A student without the algebraic background can still provide essentially the same proof. By explaining what is going on in a specific case and understanding that there was nothing special about that particular case, they can conclude that process would work the same for all cases. For example, let’s examine the middle example from above, $\frac{1}{4}=\frac{1}{5}+\frac{1}{20}$. Looking at the right-hand side of the equation, they could note that since 20 = 4 × 5, the common denominator of the right-hand side is 20 = 4 × 5; hence, we can write

$$\frac{1}{5}+\frac{1}{20}=\frac{4}{20}+\frac{1}{20}=\frac{5}{20}.$$

Note the numerator of the sum is 5 = 4 + 1, and 20 ÷ 5 = 4, so the sum reduces to $\frac{1}{4}$, as desired. The relationship between the denominators of the three fractions makes the statement work every time (**Reasoning and Proving, Connecting**). It becomes harder to justify without the compactness afforded to us by algebra, but it can be done.

We may even see **Reasoning and Proving** at the end of the problem-solving process, tied in with Pólya's step 4, looking back (**Reflecting**). In our case, we found a family of solutions where the smallest denominator in our sum was one more than the denominator of the fraction with which we started. One might wonder if there are any solutions where the smallest denominator in the sum is 2, 3, or any number higher than the denominator of the fraction with which we started. We could review our example, $\frac{1}{4}=\frac{1}{5}+\frac{1}{20}$, with our solution method in mind, and rewrite this as $\frac{5}{20}=\frac{4}{20}+\frac{1}{20}$ (**Representing, Connecting**). If we think a little deeper, we can consider this to be the equation 5 = 4 + 1, where every term is divided by 20, that is, both sides of the equation are divided by 20. If we think a bit more, we will see that there is nothing special about 20. We could have divided by anything and arrived at a true statement. For example,

$$\frac{5}{7}=\frac{4}{7}+\frac{1}{7}$$
$$\frac{5}{8}=\frac{4}{8}+\frac{1}{8}$$
$$\frac{5}{9}=\frac{4}{9}+\frac{1}{9}.$$

There is nothing too interesting going on, unless the fractions involving 5 and 4 reduce. This will only happen if we divide by a common multiple of 4 and 5, namely 20, 40, 60,… (**Reasoning and Proving, Connecting**). Doing this yields

$$\frac{5}{20} = \frac{4}{20}+\frac{1}{20} \longrightarrow \frac{1}{4} = \frac{1}{5}+\frac{1}{20}$$
$$\frac{5}{40} = \frac{4}{40}+\frac{1}{40} \longrightarrow \frac{1}{8} = \frac{1}{10}+\frac{1}{40}$$
$$\frac{5}{60} = \frac{4}{60}+\frac{1}{20} \longrightarrow \frac{1}{12} = \frac{1}{15}+\frac{1}{60},$$

which gives us equations of the desired form. Looking more closely at the new cases, we notice a couple of things (**Reasoning and Proving**):

- The two smallest denominators differ by 2 in the second case and 3 in the third.
- The largest denominator is not the product of the two smaller denominators in the second and third cases. Instead, it is their least common multiple.
- The second equation is just the first equation multiplied by $\frac{1}{2}$, and the third equation is the first equation multiplied by $\frac{1}{3}$.

We have made some other discoveries that we can examine further. We could also go beyond the natural numbers by considering fractions involving integers or real numbers as well (in this case, the explorer must be careful defining their variables to ensure that the fractions represented are defined). I will leave further playing to interested readers.

Let's now consider a new problem. The following problem also appeared in a past column (Godin, 2007):

Find a quadratic polynomial f(x)*, such that, if* n *is a positive integer consisting of the digit 5 repeated* k *times, then* f(n) *consists of the digit 5 repeated 2*k *times.*

Students have to be able to *interpret* the language and symbols to understand we are after a function f of the form $f(x) = ax^2 + bx + c$, such that $f(5) = 55$, $f(55) = 5555$, $f(555) = 555\ 555$, $f(5555) = 55\ 555\ 555$, and so on (**Reasoning and Proving, Communicating, Representing**).

Students may notice a pattern (**Reasoning and Proving, Connecting, Representing**) in the numbers involved in the problem, such as

$$\begin{aligned} 55 &= 5 \times 11 &&= (5 \times 1) \times (10 + 1) \\ 5555 &= 55 \times 101 &&= (5 \times 11) \times (100 + 1) \\ 555555 &= 555 \times 1001 &&= (5 \times 111) \times (1000 + 1). \end{aligned}$$

A bit more difficult to see is the pattern

$$1\times 9 + 1 = 10$$
$$11\times 9 + 1 = 100$$
$$111\times 9 + 1 = 1000.$$

How can this help us? By making some *inferences*, we can start to piece things together. We want our polynomial to satisfy (**Representing**):

$$f\left(\overbrace{55\ldots5}^{k5s}\right)=\overbrace{55\ldots5}^{2k5s}$$

If we let $x=\left(\overbrace{55\ldots5}^{k5s}\right)$, then $\overbrace{55\ldots5}^{2k5s} = x \times (10^k + 1)$. However, we know that $x = 5 \times \overbrace{11\ldots1}^{k1s}$, and from above, we saw that $\overbrace{11\ldots1}^{k1s} \times 9 + 1 = 10^k$, and hence, $\frac{x}{5} \times 9 + 1 = 10^k$. Thus, the function we are after is $f(x)=x\left(\left(\frac{9}{5}x+1\right)+1\right)$ (**Reasoning and Proving, Connecting**).

Re-examining our method, we can infer that replacing the 5 in our solution with a 7 (**Reflecting, Connecting**), that is, considering the function $g(x)=x\left(\left(\frac{9}{7}x+1\right)+1\right)$, we should have a function that when we input a positive integer made of only 7's, the output is an integer made of only 7's that has twice as many digits (**Reasoning and Proving**).

Let's see if we can justify this by generalizing. Let $x = \overbrace{aa\ldots a}^{k\,as}$, where $1 \le a \le 9$ is an integer, then $\overbrace{aa\ldots a}^{2k\,as} = x \times (10^k + 1)$. Thus, since $x = a \times \overbrace{11\ldots 1}^{k\,1s}$ and, as before, $\overbrace{11\ldots 1}^{k\,1s} \times 9 + 1 = 10^k$, we have $\frac{x}{a} \times 9 + 1 = 10^k$, and therefore, $f_a(x) = x\left(\left(\frac{9}{a}x+1\right)+1\right)$ is a function that satisfies

$$f_a\left(\overbrace{aa\ldots a}^{k\,as}\right) = \overbrace{aa\ldots a}^{2k\,as}.$$

Thus, f_5 was the solution to the original problem, and f_7 is the solution to the new problem involving numbers made up only of 7's.

If we dig even deeper, we can generalize this idea even further (**Reflecting**). From the original solution, and the development of f_5, we saw that for $x = \overbrace{55\ldots 5}^{k\,5s}$, then $\frac{x}{5} \times 9 + 1 = 10^k$. So, since

$$x\,(10^k+1) = \overbrace{55\ldots 5}^{2k\,5s}$$

$$x\,(10^{2k}+10^k+1) = \overbrace{55\ldots 5}^{3k\,5s}$$

$$x\,(10^{3k}+10^{2k}+10^k+1) = \overbrace{55\ldots 5}^{4k\,5s},$$

then the function $h(x) = x\left(\left(\frac{9}{5}x+1\right)^2 + \left(\frac{9}{5}x+1\right)+1\right)$ is cubic and takes in a number made up only of 5's, and the output is a number also made up only of 5's, but with *three times* as many digits. Similarly, the function $k(x) = x\left(\left(\frac{9}{5}x+1\right)^3 + \left(\frac{9}{5}x+1\right)^2 + \left(\frac{9}{5}x+1\right)+1\right)$ is quartic and takes in a number made up only of 5's, and the output is a number also made up only of 5's, but with *four times* as many digits (**Reasoning and Proving, Connecting**).

What else can we do with this? I will give you a few examples of similar functions and let the reader figure out what is going on and construct their own functional creations.

$$f(x) = \frac{2}{5}x\left(\left(\frac{9}{5}x+1\right)+1\right)$$

$$g(x) = x\left(\frac{3}{5}\left(\frac{9}{5}x+1\right)+\frac{7}{5}\right)$$

$$h(x) = \frac{x}{5}\left(\left(\frac{9}{5}x+1\right)^4 + 2\left(\frac{9}{5}x+1\right)^3 3\left(\frac{9}{5}x+1\right)^2 2\left(\frac{9}{5}x+1\right)^4 + 1\right)$$

As always, (you and) your students should be encouraged to ask those "What if…?" questions to see how much they can discover from a situation. The numbers featured in the second problem are called repdigit numbers (numbers formed by repeating a single digit). Interested readers can check out more about repdigit numbers in Godin, 2019.

I hope you have enjoyed our series highlighting some of the mathematical processes (Figure 1). I am going to make a slight change to my column going forward… no more homework! My original intention of doing homework was to give readers a chance to play with the problem themselves and then compare their thinking with mine. Let me know if you miss this feature, and I may reintroduce it.

However, with something lost, something must move in to replace it. I am hoping to get more reader input, so I am strongly encouraging readers to send me nice problems that they have found or used with their students. I am hoping to make the readers' choices of problems the highlight of the column going forward.

Figure 1: *Mathematical processes graphic from an Ontario Ministry of Education resource*

Until next time, happy problem solving!

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▲ OMCA News: June 2024

HEATHER THEIJSMEIJER
(OMCA PAST PRESIDENT)
OMCAmath@gmail.com

As another school year comes to a close, June offers members of the Ontario Mathematics Coordinators' Association (OMCA) the perfect opportunity to reflect on challenges and areas of growth. On behalf of our membership of math coordinators, consultants, and coaches from across the province, we hope this month offers you the opportunity to celebrate the successes of the past school year!

After a special five-session series with Jennifer Hlavka from The Instructional Coaching Group, our monthly virtual meetings resumed in April with guest speaker Kyne, a world-class drag queen and mathematics communicator. She shared her keynote presentation, as well as hosted a question-and-answer session, and shared excerpts from her new book, *Math in Drag*. In May, OMCA held its second in-person meeting of the year, ahead of the OAME Annual Conference. Hosted by the Limestone District School Board, members met in Kingston to engage in networking and capacity-building opportunities. Our guest speaker for that meeting was Shawn Bredin from the Ontario Ministry of Education.

This Spring, OMCA was chosen to be Ontario Teachers' Federation's representative to support the review of Ontario College of Teachers (OCT) math Additional Qualification (AQ) course guidelines. OMCA worked with other reviewers during the month of April to ensure that math AQs centre the development of math content knowledge for teaching from a stance of promoting equity and anti-oppressive education.

OMCA's final meeting of the school year in June is always devoted to conversation and exploration of issues often faced by coordinators, consultants, and coaches in mathematics education. This year, the membership will also vote in a new member of the OMCA Executive, filling the role of Vice-President. Of a celebratory nature, the winners of the OMCA Award for Excellence and Innovation in Mathematics Education will also be announced at the same meeting. Stay tuned for updates in the next issue of the *Gazette*.

Are you a math leader in your board? Interested in joining OMCA? New members are always welcome—contact us directly at OMCAmath@gmail.com, or visit our website at www.omca.website (note the different domain) to learn more about the organization and connect with us.▲

▲ In the Middle: Space's Lessons of Perspective

CARLY ZINIUK
carlyziniuk@gmail.com

Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management, and Advanced Placement Statistics at the Bishop Strachan School in Toronto. She is very active in adopting real-life data to engage her students in solving problems. Among 30 global STEM leaders, Carly is excited to have been selected for the NASA Space Apps Collective for 2023–2024. As a Canadian educator, she will be contributing toward the pilot program's chosen theme: Building an Inclusive Global Space Community.

Author's Note: I have used the guidelines as presented by *NASA's Technical Documentation Style Guide* throughout this piece when referring to the Moon, the Sun, and Earth/earth. The *NASA Technical Documentation Style Guide* capitalizes Earth when it is used as the name of the planet. In this case, it is not preceded by "the," since one would not comparably say "the Mars." If referring to ground or soil, but not the planet, then "earth" is lower case. Capitalization of "Sun" and "Moon" refers specifically to our Sun and our Moon (but lower case, for example, with "Jupiter's moons" or the moon Titan). Use of "the" with Sun and Moon is appropriate for emphasis or clarity.

"The size and age of the Cosmos are beyond ordinary human understanding. Lost somewhere between immensity and eternity is our tiny planetary home." ~ *Carl Sagan's Cosmos*

As a kid, I watched Carl Sagan's *Cosmos*, fascinated by the descriptions and images revealing the immensity of space. *Cosmos* was where I first learned about Eratosthenes' calculation of Earth's circumference. In what likely was one of my earliest "future math teacher" moments, I spent time telling everyone who would listen that Earth is not a perfect sphere. Perhaps there's a young person in your life who might be captivated, as I was, to learn that the circumference of Earth at the equator is 40 075 km, while the circumference that runs through the poles is 40 008 km, because the spinning makes it bulge at the equator!

My younger brother and I repeatedly mimicked Sagan's exaggerated emphasis on the "B" in "*B*illions upon *B*illions of stars," which was a refrain in Cosmos. In university, I learned from a physics student that Sagan had stressed the "B" so that we would not mistake it for lowly millions of stars. Sagan wanted us to hear just how immense the number of stars was, beyond our more "ordinary understanding" of millions. I now utilize a similar emphasis when teaching place value, scientific notation, and even financial math's economic comparisons, while imagining how Sagan might have created an infographic to illustrate the size of the universe for TikTok.

I watched the updated 2014 version of *Cosmos*, hosted by astrophysicist Neil DeGrasse Tyson (streaming on Disney+), and I kept noticing how frequently space exploration requires scale conversions. Just as it did as a child, *Cosmos* reignited my space curiosity, which led to my most recent work with the NASA Space Apps Collective. Throughout this new learning, I have been inspired to use astronomical examples with my math students for conversions, proportion, perspective, and scale. Here are some of the space resources that can provide your middle school students with opportunities to explore the universe and beyond, with a focus on perspective and scale.

This year, I have shown students extraordinary photos from NASA's James Webb Space Telescope (JWST) (https://science.nasa.gov/mission/webb/multimedia/images/). I labelled each photograph's distance from Earth. When I first explained to Grade 6 students that distances in space were measured in light years, their curiosity was piqued by that connection between time and distance. Admitte'dly they were confused by my description that a light year is a measure of distance, not time. I tried this explanation: a light year is the total distance light travels in a straight line during one Earth year, a legacy of Eratosthenes' early findings. My further explanations echoed my formative understanding from Sagan's *Cosmos*, and you might find this helpful to explain to students or to better understand light years yourself.

> *The dimensions of the Cosmos are so large that using familiar units of distance, such as meters or miles, chosen for their utility on Earth, would make little sense. Instead, we measure distance with the speed of light. In one second, a beam of light travels 186 000 miles, nearly 300 000 kilometers or seven times around the Earth. In eight minutes, it will travel from the Sun to the Earth. We can say the Sun is eight light minutes away. In a year, it crosses nearly ten trillion kilometers, about six trillion miles, of intervening space. That unit of length, the distance light goes in a year, is called a light year. It measures not time, but distances—enormous distances.*

The Glenn Research Centre (GRC) is NASA's Zero Gravity Research Facility, the largest in the world. GRC has produced a series of Mathematical Thinking activities, many of them space-informed Fermi-like problems like, "How far is a light year?" Another approach is to show the NASASolarSystem video (youtu.be/MX3PIkbTQwQ), *Our Milky Way Galaxy: How Big Is Space?*, and ask the students to generate mathematical questions of scale. NASA's Jet Propulsion Lab (JPL) has a related classroom activity intended for younger students, modelling the distance to space by stacking coins or washers on a local scaled map and asking the students to explore the following:

- What local landmarks are within a 100 km scaled radius on your local map?
- How far does that actually measure on your map?
- How does that compare to the altitude of some key astronomical features?
- What would a stack of coins or washers look like for this distance?

	Approximate distance above Earth (in km)	How many coins above Earth?
Von Karman Line (where astronauts enter "space")	100	
International Space Station	400	
Hubble Space Telescope	525	
Canada's RadarSat	798	
The Moon	384 400	
JWST (2nd Lagrange Point, L2)	1.5×10^6	

Astronomer Andrew Fraknoi (www.fraknoi.com) has widely presented several activities for students, educators, and parents/guardians to explore astrotourism, understand the size and distances of constellations, and debunk astrology. The following printable activities would be easily adaptable for middle school students.

- Where will Bill Gates' Great-Great Granddaughter Go for Her Honeymoon? The Cosmic Tourist Sights Activity (You could consider Astronaut Jenni Sidey-Gibbons as a Canadian alternative to Microsoft's founder!)
- Cosmic Calendar (Scale of the history of the universe)
- How High Up Is Space? (Where space begins and modelling scale)

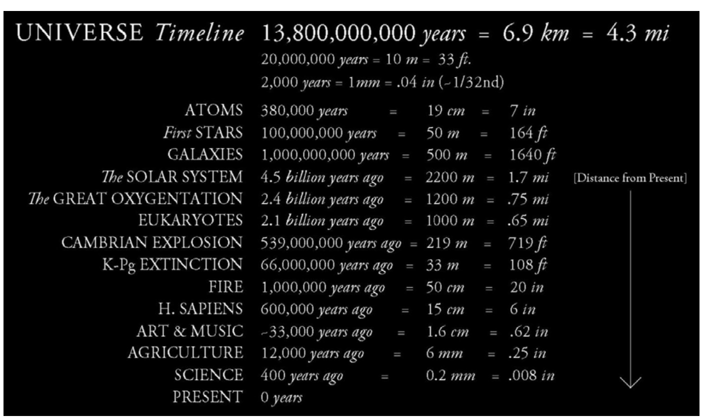
Fraknoi and JPL both encourage learners to construct a physical solar system scale. The JPL activity, *Solar System*

Scroll, uses paper rolls (used for payment machines, available at an office supply store) or toilet paper, to predict the distance between the planets, and then calculate to check their estimations. We can also use this as a connection to fractions as below. As with the *How Big Is Space?* video, NASAJPLEdu's video (youtu.be/DMZ5WFRbSTc), *Solar System Size and Distance*, also provides many scale references. These modern videos evoke Eames' 1977 *Powers of Ten* film. I have regularly encouraged teachers working with logarithmic and exponential functions to use these videos to understand the value of logarithmic scale.

Figure 1: *Solar System Scroll from NASA's Jet Propulsion Laboratory's Classroom Activity*

The *To Scale* series (www.toscaleseries.com/time), with the tagline "Shift Your Perspective," helps students to understand the relative scales in a Universal Timeline. *To Scale: Time* is the short film (youtu.be/nOVvEbH2GC0), which illuminates this data, showing "13.8 billion years of cosmic evolution, and our place within it." Data lovers rejoice!

A practical scale model of time:



UNIVERSE *Timeline*	13,800,000,000 *years*	=	6.9 *km*	=	4.3 *mi*
	20,000,000 *years* = 10 *m* = 33 *ft.*				
	2,000 *years* = 1*mm* = .04 *in* (~1/32nd)				
ATOMS	380,000 *years*	=	19 *cm*	=	7 *in*
First STARS	100,000,000 *years*	=	50 *m*	=	164 *ft*
GALAXIES	1,000,000,000 *years*	=	500 *m*	=	1640 *ft*
The SOLAR SYSTEM	4.5 *billion years ago*	=	2200 *m*	=	1.7 *mi* [Distance from Present]
The GREAT OXYGENTATION	2.4 *billion years ago*	=	1200 *m*	=	.75 *mi*
EUKARYOTES	2.1 *billion years ago*	=	1000 *m*	=	.65 *mi*
CAMBRIAN EXPLOSION	539,000,000 *years ago*	=	219 *m*	=	719 *ft*
K-Pg EXTINCTION	66,000,000 *years ago*	=	33 *m*	=	108 *ft*
FIRE	1,000,000 *years ago*	=	50 *cm*	=	20 *in*
H. SAPIENS	600,000 *years ago*	=	15 *cm*	=	6 *in*
ART & MUSIC	~33,000 *years ago*	=	1.6 *cm*	=	.62 *in*
AGRICULTURE	12,000 *years ago*	=	6 *mm*	=	.25 *in*
SCIENCE	400 *years ago*	=	0.2 *mm*	=	.008 *in*
PRESENT	0 *years*				

Figure 2: To Scale: Time *timeline scales and distances*

I encourage you to check out the other scientific data sets and short films in the series: *To Scale: The Atom, To Scale: The Solar System, To Scale: The Stars, To Scale: The Galaxy,* and *To Scale: The Cosmos*. Similarly, Josh Worth has created a "tediously accurate model of the solar system" (joshworth.com/dev/pixelspace/pixelspace_solarsystem.html) as an interactive web page, *If the Moon Were Only a Pixel*, which elevates the JPL paper-roll activity (pun intended).

As I have showcased previously in this column, *SpaceMath@NASA*, from the Chandra X-Ray Observatory, has a set of classroom-ready math activities initiated by astronomical applications. Many can also be used as extensions or independent challenges for Grades 7–9 students. These problems are authentic glimpses of modern engineering when designing satellites, investigating space, or exploring weather and climate. Since teachers recommended this format, the *SpaceMath@NASA* activities are compact "one-pagers" with a student worksheet (complete with a top line for the student's name) and corresponding Teacher's Guide and Answer Key page. I highlight the following activities from *SpaceMath@NASA* that focus on scale, perspective, and conversion problems:

Problem 300: Earth's Rotation Changes and the Length of the Day? Students use tabulated data for the number of days in a year, from 900 million years ago to the present, to estimate the rate at which an Earth day has changed, using a linear model. [*Highlighted topics: graphing; finding slopes; forecasting*]

Problem 398: The Crab Nebula—Exploring a Pulsar Up Close! Students examine a photograph of a pulsar, and use scale and time taken by light and matter to reach a specific distance. [*Highlighted topics: scale drawings; unit conversion; distance = speed x time*]

Problem 538: How Big Is Our Solar System? Students convert solar system distances into Astronomical Units. [*Highlighted topics: proportions; unit conversions*]

Problem 539: Visiting the Planets at the Speed of Light A continuation of the previous activity, students then can transfer the Astronomical Units to travel times, given the speed of light. [*Highlighted topics: proportions; unit conversions; time = distance/speed; metric units*]

SpaceMath@NASA also contains Grades 6–8 curriculum modules outlining a teaching sequence that can be used for entire courses, specific units, or individual lessons. These modules are designed around the 5E Constructivist Learning Cycle. The Proportional Reasoning Module for Grade 6 presents a NASA Press Release about the number of stars in the Milky Way and a corresponding NASA eClips video. Students use proportions and fractions to investigate these NASA calculations, then compare the relative sizes of stars to our own Sun.

Further in the *SpaceMath@NASA* series, the Image *Scale Math NASA Book* provides activities for Image

Scaling. Astronomers use image scaling to understand any image-type data produced by telescopes or satellites, in the sky or on the ground. Each *Image Scale* activity includes a solar system image and an indication of the physical field of view in m or km. Students then measure components of the solar system photo in mm, and scale the image measurement by m per mm, or km per mm. They investigate features in the photograph to determine their actual physical size. One example, taken by Mars Reconnaissance Orbiter, shows a crater wall in Mars' southern hemisphere from an altitude of 450 kilometers. You can see evidence of water flowing downhill from the top left to the lower right.

Figure 3: *"Water on Mars" photograph from NASA/JPL/Main Space Science Systems, Image Scale Math, p. 6*

Figure 4: *Downloadable background for NASA Pi Day Challenge 2024*

NASA's JPL has annually produced the *NASA Pi Day Challenge*, which includes posters, backgrounds, articles ("18 Ways NASA Uses Pi," "How Many Decimals of Pi Do We Really Need?"), printable student worksheets with illustrated answer sheets, and further connections with information about the space mission/astronomy materials related to each problem for extension. The 2024 edition, the eleventh year of NASA creating this challenge for educators and students, includes problems about a cat, "Taters," chasing a laser, NASA crashing a spacecraft into a wayward asteroid, and a satellite photographing Earth and how long it takes to spot differences on our planet's surface.

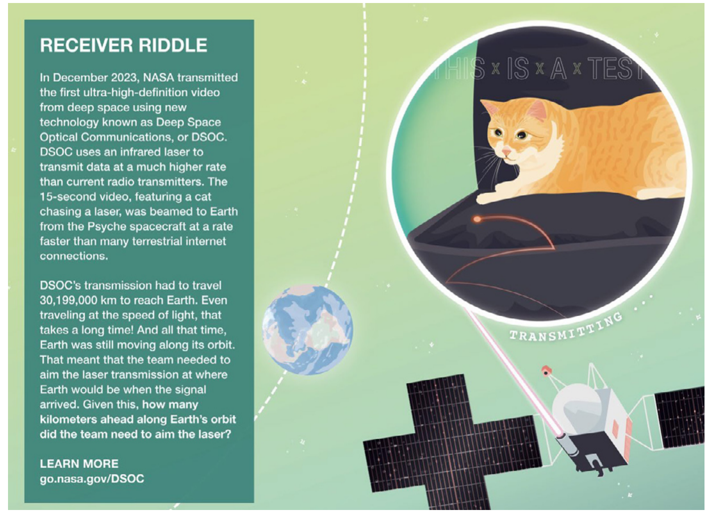


Figure 5: *2024 NASA/JPL Pi Day Challenge Problem ~ Receiver Riddle*

JPL also creates STEM and math-specific activities that connect to NASA Press Releases, labelled "Teachable Moments." These activities also include a Student Worksheet and a Teacher Guide (both in downloadable pdf form), assessment, and extension opportunities. Here are two sample activities from the JPL Education-Teach STEM Resources:

Sizing Up Pluto, and related July 16, 2015 Teachable Moment, *NASA's New Horizons Mission Flies by Pluto*. Example:

> *On July 14, 2015, NASA's New Horizons spacecraft sped past Pluto—a destination that took nearly nine and a half years to reach—and collected scientific data along with images of the dwarf planet. Through careful measurements of new images, scientists determined that Pluto is actually larger than previously thought: 2370 kilometers in diameter. Because of the orbital interactions between Pluto and its moon, Charon, Pluto's mass is well known and understood (1.31 x 10^{22} kg). Having a more precise diameter gives scientists the ability to more accurately calculate Pluto's average density. This is important information for scientists because it helps them understand the composition of Pluto.*

Students use these measurements to determine the volume, density, and circumference of Pluto, and compare these to the previous measurements.

Math Rocks: A Lesson in Asteroid Dynamics related to the April 19, 2017 Teachable Moment, *How NASA Studies and Tracks Asteroids Near and Far*. Example:

> *On Feb. 15, 2013, a small asteroid entered Earth's atmosphere over Chelyabinsk, Russia, and startled onlookers with its fiery appearance and shockingly loud noise. NASA estimates the asteroid was approximately 17 meters in diameter, with a mass of approximately 11 000 metric tons, and travelled approximately 18 kilometers per second. Infrasound stations nearby and as far away as Antarctica, and nearly half-way around the world, detected the low-frequency sound waves generated by the meteor. The infrasound data indicates that the event, from atmospheric entry to the meteor's airborne disintegration, took 32.5 seconds. The entry angle to horizontal was about 15°, and the terminal part of the fireball was at about 20 km altitude.*

Students use this data to determine the straight-line path, the altitude, and the density, and compare this asteroid to other substances on Earth.

Astromaterials Research & Exploration Science (ARES) is the physical science research department at the Johnson Space Center, which curates all NASA-held extraterrestrial samples! ARES also provides additional hands-on inquiry-based STEM activities. *Oh, What a Pane!* provides a mathematical scaling approach to investigating windows on Earth and in space. Students collect and analyze data from astronaut-taken photographs, many taken from the cupola on the International Space Station (ISS). After reviewing this ARES activity, I showed my students this photo of Jessica Meir's feet in the cupola of the ISS and asked, "Can they use the size of their own feet to approximate the measurements of the cupola?"

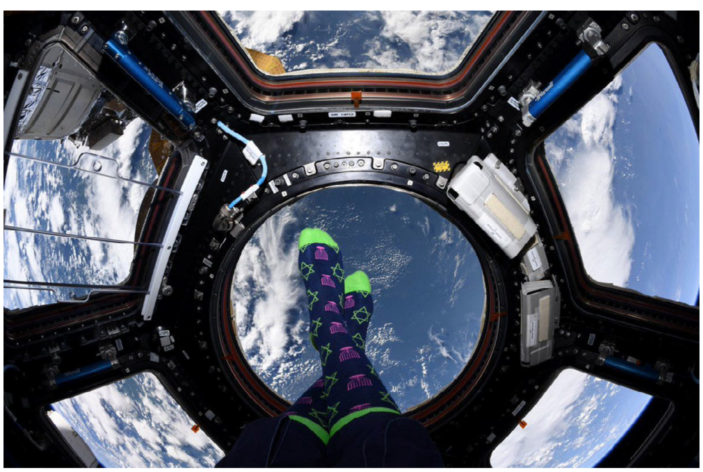


Figure 6: *Photo from Astronaut Jessica Meir @Astro_Jessica Dec 22, 2019*

NASA's *Image of the Day* can be used as a class starter to communicate scale and perspective. Most students have not viewed Earth from the perspective of the cupola on the ISS, with or without Hanukkah-themed socks. Using images and space-scale calculations can help students to understand why many of the seemingly isolated skills we teach in math are so essential to anyone truly curious about space. Whether that is calculating just how far away the Pillars of Creation formation is, or exploring the billions and billions of stars through a full-resolution picture of Stephan's Quintet from JWST, students rarely have to be coaxed to use scale or conversion because its relevance is immediately apparent. I hope this has given you and your students many new perspectives on how to use space exploration in your classrooms!

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Out and About

The $O_3$4ME Spring Social

Mac^2 Group at the Annual Conference

Geometry Found at the Annual Conference (St. Lawrence College)

▲ Elementary Math Matters: Talented, the Overlooked Special Need

LYNDA COLGAN
lynda.colgan@queensu.ca

Lynda Colgan is Professor Emerita, Faculty of Education, Queen's University. Lynda is currently working on a number of research, resource-creation, and teacher-education projects, funded by the National Sciences and Engineering Research Council of Canada (NSERC) and the Social Sciences & Humanities Research Council (SSHRC).) She will carry on her long-established commitment to STEM outreach by continuing to seek out creative avenues to engage students, parents, and educators in mathematics at home and in the classroom.

During my Margaret Sinclair Memorial Award lecture at the Fields Institute on September 23, 2023, I recounted a number of memorable people and experiences that shaped my life and career as a mathematics educator and Science, Technology, Engineering, and Mathematics (STEM) ambassador. They are a good place to begin this column.

At Bruce Public School, there was Mrs. Patricia Harvey, who encouraged a small group to learn about calculations in bases other than 10 and let us "skip" class to watch science history in the making: John Glenn's 1962 launch into space (on a tiny black-and-white, rabbit-eared television in the staff room), making him the first American to orbit the Earth.

There was Mr. Pruder, the Visual Arts teacher from Woodfield Road Senior Public School, who acknowledged the terror some of us suffered when confronted with a blank page and the challenge of "drawing." He allayed our concerns by teaching us how to apply perspective, ratios, and strategies, such as the rule of thirds, to our creations.

At Riverdale Collegiate Institute (RCI), there was Ms. Patricia Hall. In Grade 10, every Saturday, a small group of awkward teens took the Queen Street streetcar and University subway to the stop at Queen's Park, then walked a block west for our class in a room at the Toronto Board of Education office. We bubbled in computer programs on small stacks of cards that would be run on a mainframe computer; the huge sheets of sprocket-holed paper returned days later for debugging in advance of the next Saturday. It took me three weeks to successfully run a program that counted to 10.

At RCI's "mathematics club," we honed our problem-solving skills for the Pascal, Cayley, Fermat, and Euclid contests—a tightknit group of competitive students, looking for a challenge and happy to be among "our people"—the chalkboard-scribbling contingent.

In Grade 12, there was Alex Trebek (yes, *that* Alex Trebek). Anyone familiar with *Jeopardy* can visualize his understated, but bemused, expression when he asked me to, "Pick one, Lynda," as I waffled between "more" and "less" too many times in response to the question, *Is ice more or less dense than water?* You could hear the collective, but good-natured, groan of the RCI *Reach for the Top* team's cheering section when I finally did pick.

In Grade 13, I was the first student in the "Hallway" (the official, collective noun for a group of chemists) to have her debut as an organic chemistry magician (under the watchful supervision of teacher Mr. Coutts), performing a "trick" with two colourless solutions—one of which floated on top of the other (a solution of decanedioyl dichloride in cyclohexane floated on an aqueous solution of 1,6 diaminohexane, to be precise). The class was amazed as a long polymer thread formed at the interface. I pulled it out as fast as it was produced, and with an exaggerated "Ta Da!" I explained that I had used the magic of chemistry to make a "nylon rope."

There were many more opportunities, but space does not permit their enumeration here. I was fortunate to have been challenged as needed, and guided through enriching learning experiences by educators who were simultaneously gifted in differentiating their teaching as well as caring.

The Post-Lecture Post-Mortem

After the lecture, my colleague, Dr. Ann Kajander, an almost lifelong Thunder Bay resident, contacted me to regale me with her similar experiences. High on her list were the special Saturday sessions for students who had achieved high scores in the University of Waterloo competitions. They spent the day at Lakehead University, attending a series of designed-with-them-in-mind lectures by members of the mathematics department, followed by a celebratory pizza supper. She also spoke of special opportunities to visit the computers at Lakehead and experiment with programming. The conversation soon turned to the genesis of her early books, *Big Ideas for Small Mathematicians: Kids Discovering the Beauty of Math* and *Big Ideas for Growing Mathematicians: Exploring Elementary Math.* A mother of two bright and curious young boys, who were "bored" by their Kindergarten program, she found that other parents shared her frustration with the decidedly pedantic activities provided by the teachers at the time, and took it upon herself to teach a clutch of Kindergartners about fractions, Möbius

Strips, and geometric transformations, using suncatchers, ant paths, and kaleidoscopes in both her home and community.

Those reminiscences resulted in Ann and me having a lengthy conversation about how the current concentration of attention and resources to those mathematics students who are at risk of not earning their math course credit, or by virtue of the Provincial Standard used with EQAO testing, has left children who need enrichment all but forgotten. Why, we queried, is it that policy-makers see providing data-proven remediation for mathematically challenged learners as a priority, but have no issue in ignoring the equally pressing need to provide essential fuel to hungry learners?

We spoke of the sad demise of the Provincial Mathematics Olympics for students in Grades 7 and 8, the dearth of local chapter competitions, and the absence of motivation given the absence of a provincial title. We talked about how diminishing budgets are impacting schools' decisions to reduce or eliminate participation in the University of Waterloo mathematics and computing competitions. We compared notes on the limited or non-existent availability of enrichment courses for academically motivated students, and gifted programs for formally identified students in our catchment areas. We did not need both hands to count the number of pan-Canadian post-secondary mathematics departments that offer summer camps or enrichment programs intended for high school students, and found our tally was greatly exceeded by the number of post-secondary institutions that have closed their Summer Camp programs and Enrichment Studies Departments.

According to the Canadian Science Policy Centre[1], outreach efforts by post-secondary institutions do not provide enough emphasis on the entire academic trajectory of an individual. In other words, most institutional STEM action plans do not address the pervasive and systemic roots of inequity experienced by students from under-represented groups at earlier stages in their STEM education, specifically elementary and secondary school. We agreed that among the "under-represented" are those students who demonstrate both interest in and aptitude for learning.

Through our dialogue, Dr. Kajander and I agreed that from our perspectives, gifted and talented students are treated inequitably, and many harmful myths about them continue to persist.[2] For example, it is widely believed that gifted students will succeed on their own, learning despite the context and in spite of the teacher; always at the "top of the class" grade-wise. In fact, many talented students are "twice-exceptional" in that they may be both gifted and dyslexic—a reality shared by five Nobel laureates.[3] There is widespread responsibility foisted on gifted students in mainstream programs to be role models and peer tutors/mentors for other classmates (some of whom disparage the "smart" students outside of class[4]): the myth being that the students who "get the math" enjoy those roles and the attention that they bring, when the truth is that some do not like being surrogate teachers. The final insult is that the media and the public have loudly argued that the term "giftedness" is exclusionary and elitist[5]—the purview of the privileged. In fact, "gifted" learners are found in all cultures, ethnic backgrounds, and socio-economic groups—it is the specialized programs for them that are exclusionary and elitist because they are not equitably diffused across sectors by administrators and policy-makers.

To Identify or Not…

While "giftedness" is a special education category, it is not a disorder in the *Diagnostic and Statistical Manual of Mental Disorders*. Nonetheless, "giftedness" can only be designated through testing by either a physician or specially certified psychologist. There are many masterful mathematics students, who have not been formally "diagnosed," and many more who may, in fact, not make the cut-off on highly standardized and rigorous psychometric evaluations, but those students are still exceptionally capable. As a result, students who are struggling with mathematics and students who excel in mathematics find themselves in the very mixed achievement demographic typical of most regular classrooms, including destreamed secondary school mathematics classes. Unfortunately, research[6] has found that nearly 80 percent of teachers of "highly achieving students" paid those children the **least** attention, leading to "minimal" academic gains compared to their academically struggling peers. In the same study, 86 percent of teachers responded that in the spirit of justice and equality, their attention must be given "equally to all students, regardless of their backgrounds or achievement levels." The contradiction between belief and practice is concomitantly dramatic and discriminatory.

At the heart of the controversy around "giftedness" are the negative connotations correlated to the label. It has long been argued that *labelling* (rather than identifying) any child may lead to stigma. For example, the descriptor "special needs" evokes deficit-model assumptions and images of physical, cognitive, or mental disabilities. A direct consequence of such a label may be that the child so categorized or characterized may not only face social barriers, stereotyping, discrimination, and negative judgments from peers and adults, but fall prey to a self-fulfilling prophesy of failure and defeat. By "tagging" highly able children as "gifted," people (especially the children themselves) may equate the label as meaning "better," which in turn may cause the students to mask their socio-emotional and learning needs, and their teachers to overlook them.[7]

It is obvious that neither the public education system, nor parents, can afford to have every child who has a hunger for learning to undergo the extensive standardized abilities-testing that is required for the "gifted" designation. It must be acknowledged, however, that the same delayed or missed identifications that leave special needs students without access to early interventions is equally true for talented students who require enrichment. And whether one likes labelling or not, without them, teachers are left without a clear understanding of how to support a special needs student. Perhaps most importantly, that student is left without the self-knowledge that is key in becoming an advocate for their own needs. The potential for significant and negative impact on a student's academic success and educational outcomes is clear.

Formally diagnosed or not, we must recognize that there is a significant subset of students who demonstrate a willingness and desire to do and learn more in mathematics, and those students deserve the dignity and respect given to other neurodiverse children. The focus of the socio-emotional side of mathematics learning has tipped the scales decidedly toward efforts to reduce negative feelings caused by poor mathematics achievement and test anxiety. Yet research tells us that "gifted" and talented students also face an increased risk of anxiety, depression, and low self-esteem, along with social and academic problems caused by their portrayals as nerds or misfits. Research also tells us that many in this subset of students are underachievers: inattentive, unengaged, and chronically bored[8]… surely a strong argument for enrichment as educational intervention.

Giving All Students—Including Gifted Students—a Chance

So, what do we do? We step up the game in our elementary and secondary schools, just as my teachers and some university professors did for Ann and me so many years ago. Our teachers differentiated learning before differentiation was a "thing" by introducing the entire class to fascinating people, places, and ideas, and then providing creative and productive outlets for those of us who "bit." They directed us on the basis of their observations and qualitative evaluations to local mini-courses and mentors of interest, and allowed us to be self-directed and independent learners based on their knowledge of us personally and individually. Perhaps, most importantly, they brokered environments and opportunities populated by students with whom we fit in just fine—where we felt a sense of belonging and satisfaction. Our teachers deeply understood that no classroom was homogenous, so they also quietly, but purposefully, directed some classmates (including some from our group) to equally important painting, music, dance, language, and athletic opportunities—wanting us to explore possibilities within and outside of the classroom. My teachers operated without the strict definitions, rules, and criteria that seem to be reducing, rather than increasing, opportunities for all, but especially gifted students in this current era of diversity and inclusion.

If I were to imagine a perfect learning situation for every advanced learner, it would not be one in which every student gets a ribbon for "showing up and competing," but one in which we provide the intellectual and physical playgrounds that celebrate difference, without loss of dignity for those with the potential to "win big," whatever the context, by honestly acknowledging the real-world reality that:

> *...we tune in to sporting events to watch gifted athletes perform amazing feats for our teams. We buy the new inventions created by the gifted that make our lives easier and richer. We read the books and watch the movies that emerged from the minds of those who have a natural talent for story-telling.*[9]

Surely if we hold in the highest regard the gifted athletes, musicians, artists, writers who bring joy and thrills to our lives, and appreciate the engineers, physicians, astronauts, computer scientists, and mathematicians for their positive contributions to our quality of life, we must recognize that such talents do not develop in a vacuum, nor at the end of one's public education. Extremely capable mathematics students have the right to dedicated enrichment interventions from specially trained educators throughout their formative years of elementary and secondary education, if they are to achieve self-fulfillment, and if society is to reap the benefits of their unique competencies.

Notes

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Continued on Page 35

▲ Coding in the Classroom: What Does It All Mean?

IAIN BRODIE
iainbro1.625@gmail.com

Iain has been an elementary school educator for 28 years, and is now into a second career, teaching in-service and pre-service teachers. He teaches face-to-face at Ontario Tech University and virtually at Western University.

HATICE BEYZA SEZER
hsezer2@uwo.ca

Beyza is a former teacher and long-term researcher. She is currently a graduate student in education at Western University, and her research focus is the integration of computational thinking into mathematics education.

(Quotations within this column may have been lightly edited to reflect the difference between oral and written language.)

Since September, our columns have delved into the murky waters of thinking about what kind of thinking is going on with students as they learn to code and begin to learn through coding. Our journey has mirrored our inner struggles between theory and practice, and ends up focusing on how we might support teachers in the classroom. At a certain point during a symposium on computational modelling, our struggles came to a head as we participated in a group that wrestled with different models of computational thinking (see Figure 1). We stepped in like the Lorax did, and spoke for the teachers, asking about the value proposition for implementing coding in their classrooms.

What does it all mean to begin coding with our students? Why should we do it? Is the value of what our students will get from it equal to or greater than the value of our time invested in learning to implement it?

So we took these questions, and more, to a teacher implementing coding in his classroom and found out that all the wrestling with theory and practice, all the debates about the types of thinking involved in coding, do have value for our practice, as well as preparing the way for what is needed next, if Ontario really wants all of its students to be able to learn through coding.

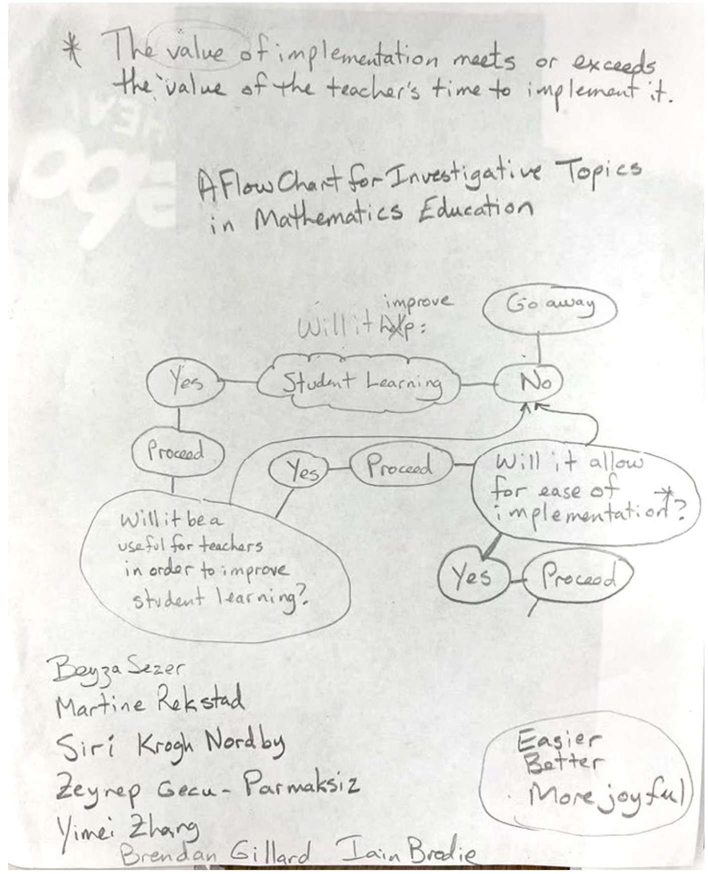


Figure 1: *Computational thinking investigative topics sieve, developed at the Mathematics, Computational Modelling and Equity Symposium, Spring 2023*

Meet Adam Leyland, a Grade 4 teacher in Oshawa, Ontario. An experienced teacher in his second career, he comes to education with a background in network engineering, where he developed and programmed networks for a major telecommunications company. In other words, he knows how to program, but still needs help with how to teach coding to his students. Throughout our conversation with Adam, we noted that he and his students are indeed showing some of the ideas behind many of the different conceptions of the types of thinking we have been looking at. Besides our conversation for this column, Iain has been in Adam's classroom, gleefully teaching his students to code, and some of his observations are included in the ideas presented in this column.

Computational Participation?

The first thing we discussed was the notion that students today are more technologically advanced than in the past. "They talk about children as being the most technologically advanced generation we've ever had," Adam said, "which is a bit of a misnomer because they're the most, you know, swipe and stream and look-at-YouTube generation that we've ever had." This is an important point for anyone teaching coding in their classrooms. Students today are, by and large, consumers of technology, not masters of it. That we are bringing them the inner workings of some technology and the processes involved in creating it is an important part of their education and making them aware of the potentials and limitations of computational models. This awareness is part of what the proponents of computational participation, like Resnick and Kafai, have to say. At the very least, learning to code will allow our students to become more informed and more able to think critically about technology in the future.

Computational Thinking?

Students do not generally like to plan, or brainstorm. When engaging in problem-solving, the "understanding the problem" part of the process is frequently rushed through to get at trying a solution, even if, in retrospect, there wasn't full understanding and it did not make sense. Mathematicians spend the majority of their time developing understanding with problems, but not students. At some point in their education, students who become mathematicians must learn to slow down and think.

This is a problem in all subjects, not just mathematics. Adam uses the example of programming and errors to show how coding can act as a kind of speed bump to teach students to slow down their thinking.

I think an even better visual is using the coding to make a little robot, like an mBot or a Sphero, do something very specific. You've told it what to do, and the easiest way to see where you're coding might have been wrong, or where you didn't think far enough ahead with your coding is by watching the robot do the wrong thing once you've programmed it.

Through experience, students can learn to think ahead, to avoid known pitfalls. They can come to learn to think before they do. This is an important skill for many different subjects. In writing, the most important phases are brainstorming and revising. This is where most good writers spend the majority of their time. This process is built into programming and coding.

Writing has a lot of different nuances to it, whereas in coding, if you jump over those steps, you make an error and your end result is not what you wanted it to be. So, the idea is—to me—that's also, again, the planning which goes into science and language. And I'm sure you can use it even in math. That planning? It's so much more acute in coding than it is in any other curriculum subject, in my mind anyway. So, it is yet another avenue you're teaching kids to sort of measure twice and cut once, or to plan it before you do it.

This is almost exactly what Janette Wing had to say about the process of computational thinking back in 2006.

Computational Fluency?

Then there is the idea that everyone needs to become more computationally fluent in order to become a fully participating member of society. To this, Adam gives the example of not being left behind technologically where, "You're trying to expose kids to something that they're not likely to see in their childhood until they get into adulthood. And now there are careers that are open to them, specifically with coding or just, again, the technology that they will inherit 15, 20 years from now will be vastly different from the technology they're sitting with now." So even if we are not preparing our students to become software engineers—we're not—what we teach them throughout their K–12 education will help to make them better able to handle the technological change that is coming.

Computational Opportunity?

We asked Adam what he would want to help him code in his classroom, if money were no object. He made the point that teachers and students are in fact technologically adept, but that they are not necessarily technologically knowledgeable enough to know how to proceed in their classrooms. He would love to see some teachers hired, who could go around a few schools at a time and work in depth with the classroom teachers. They would get teachers started, move on to a new school, while the teachers worked on it with their students. Next, they would return to consolidate and move on to the next step continuing the process, three-part lessons for the practising teacher.

Knowing this is not probable, Adam asked for a practical resource, or set of resources, that would show him what to do and then what to do next. Not so much *how* to do it; he has the pedagogical knowledge and, after being shown once, doesn't need that anymore. Not why to do it, but *what*. What is possible? What are the mathematical topics that lend themselves well to computation? Are there ways of coding in subjects other than mathematics and science, and what does that look like?

If anyone out there would relish the opportunity to create something and make a difference, there is an appetite out there for just this kind of thing. In Adam's school alone, quite a few teachers are eager for someone to help guide them. And as Adam says, "It's not about, well, now I don't have to teach coding. It's 'fantastic. Someone will actually show me how to do this.'"

What It Is

So, is it computational thinking? Modelling? Participation? Fluency? Literacy? A computational tool?

Yes.

All of them.

Each of them has something to help us think about why, how, and what we should be doing computationally in our classrooms. Keep discussing what you think the thinking going on is while coding, but please, we speak for the teachers, move quickly to thinking about what teachers might code in their classrooms and how we might get that information to them effectively.

Adam had much more to say than what we covered in this column. If you would like to hear more, head over to OAME Talks and listen to Coding in the Classroom 11. ▲

OAME/AOEM is starting the process of looking for a new Executive Director(s). The position would begin Sept. 1, 2025. The position includes, but is not limited to:

- managing all correspondence related to OAME/AOEM
- maintaining the membership records: reminding when expiry dates are looming, adding new members or information, sending receipts
- managing finances and budget of the organization from year to year and working with the accounting firm to prepare annual statements
- arranging for the printing and mailing of the *Ontario Mathematics Gazette* quarterly
- organizing and helping to plan all meetings through the course of the year
- being a liaison with OAME/AOEM's local chapters and board committees
- assisting with visioning and forward planning of the organization
- attending annual conferences and staffing a table in the sponsors area

The successful candidate(s) will be provided with training during a transition period.

Anyone interested in being considered for the position should contact Eds@oame.on.ca.

▲ Book Review: A Teen Perspective on Financial Literacy: A Review of *A Rich Future*

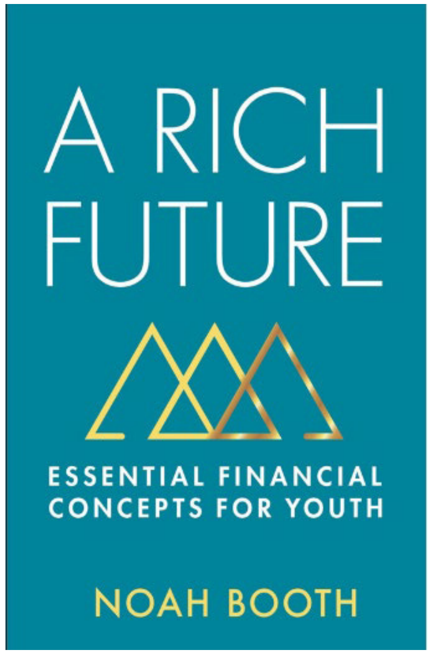
TIM SIBBALD
timothys@nipissingu.ca

Booth, N. (2021). *A rich future: Essential financial concepts for youth.* Government of Canada. www.richfuturebook.com/

It is rare that one finds a teen selling books in the exhibitor area of the OAME Annual Conference. Even more rare is to discover that the teen wrote the book that they are selling. This is exactly how I came upon Noah Booth and his book about financial matters. As an astute entrepreneur, Noah provided the *Gazette* with a free copy, on the understanding that it would be reviewed (and I owe him an apology for being slow getting it done).

The book is interesting on multiple levels. For students, it could be useful as a form of textbook to augment classroom instruction. For teachers, it provides a youthful perspective that may give pause to consider examples they use, language for conveying ideas, and the way the times have changed with respect to some choices regarding finance.

As a potential textbook, this resource serves as having many short readings that will not overwhelm students. Most can serve as starting points for classroom discussion and launchpads for teachers to elaborate. It is not a stretch to have students discuss ideas from assigned readings as a minds-on activity leading into instruction that can further develop the ideas. The benefit is that many concepts are explained in teen-friendly language, and in a way, that may reflect the sort of interests and perspectives that teens have. There are a few places where the author’s interests are



emphasized (the example of purchasing a gym membership that assumes one will actually use the gym, for instance), but that is okay because it gives a personal sense to the book. Some sections later in the book may not resonate with students, as they seem to reflect advice the author has received, rather than experienced; however, these are a small portion of the book and can serve to keep student readers on their toes. (Note that the website for the book mentions lesson plans that I am not reviewing.)

For teachers, many ideas in the book may cause reflection on how the times have changed. The sense that youth may simply move to investing without using fixed income tools, such as bank accounts, is likely a sign of the times. It is explained in terms of inflation generally being higher than bank interest rates, and the clear access to investing tools in the modern age. The rationale is provided in a reasoned manner, though there may be some points that are useful for discussion with students. In this respect, excerpts or quoting the book could be used as a conversation point.

The book has a few small weaknesses that an instructor can fix. There is a general assumption that university or college is better than not pursuing either, and this suggests that parts of the book (p. 57) may not be as effective in Essential-level courses. The only error I noticed is the claim that credit cards charge interest on the unpaid balance (pp. 20 and 96). It is my understanding, though I lack personal experience, that it is a little more complicated. If the balance is not paid off in full, then interest is charged from the day of the purchase, and continues to be charged on the balance remaining. For example, if you use a credit card to purchase $1000 on June 10, and you pay off $200 on the billing day of June 25, there will be interest charged on the full $1000 for 15 days and then interest on the $800 balance in the next month. (See www.nerdwallet.com/ca/credit-cards/credit-card-grace-period.)

It is suggested that students can play the “long game” (p. 73), which suggests instruction will need to address removing money from investments. There is a rosy view that one buys into investments and sees the value go up; however, as experienced investors, many teachers will be aware that if you need money at a particular time, then disposal of an investment to avoid loss of money is an important detail. The sense of the long game is surprising when paying for higher education is clearly in the author’s sights. A related issue is that the author hedges when suggesting between investing in Exchange Traded Funds (ETFs) and indexed funds, as opposed to stocks. He seems to realize that many teens will not want to do the research he recommends for stocks, and that ETFs are a compromise.

Toward the end of the book, the topic of RRSPs is mentioned. They are mentioned after Registered Education Savings Plans (RESPs) and Tax-Free Savings Accounts (TFSAs). While the order of investing is reasonable, the author does not appear to be aware of the consequences of placing money in an RRSP if you anticipate your income is going to increase and you will wish to use the money that is placed in the RRSP. He mentions the tax consequences, but not the potential impact of withdrawing money after one’s salary has risen.

An astute reader will notice that the issues I found with the book are very specific. The overall impact of the book is its key strength, and it is well suited to a classroom resource, where every student has a copy. My purpose in being specific is that I am focusing on implementation because the teen voice is clearly a beneficial addition to instruction in financially oriented math classes. ▲

Continued from Page 31

4 Gordon, S. (2021). Why gifted students are targeted by bullies. *VeryWellFamily*. www.verywellfamily.com/how-bullying-impacts-the-gifted-student-460594#:~:text=Because%20gifted%20children%20are%20often,something%20is%20wrong%20with%20them

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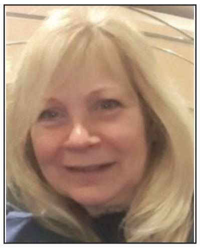
▲ The Ongoing Challenge of Assessing Thinking for All Learners: Examples of Evaluation Tasks for Grade 9



ANN KAJANDER
akajande@lakeheadu.ca

Ann Kajander is a professor of mathematics education at Lakehead University, as well as an experienced secondary mathematics teacher. Her recent work concentrates on teacher development in mathematics, with a focus on the specialized content knowledge domains of representation and reasoning. Teaching Intermediate Mathematics: From Models to Methods *is her most recent book.*

WALID KHNEISSER
Walidkhn675@gmail.com

Walid Khneisser is a recent graduate of Lakehead University. He advocates for the use of thinking classrooms over traditional classrooms, to foster knowledge and fundamental understanding of mathematical concepts. He believes in supporting growth in mathematical understanding, rather than the rote application of formulas. Walid Khneisser is also fascinated by learning about using visual manipulatives to make sense of abstract mathematical concepts.

With the transition to the Grade 9 destreamed Mathematics course in Ontario now in full swing, Ontario teachers are continuing to work to authentically enact the vision of the new course (McDougall et al., 2023).

One area that is always especially challenging during phases of curriculum implementation is the revisioning and redesigning of assessment and evaluation methods. The *Growing Success* document (Ontario Ministry of Education [OME], 2010) reminds us that "the primary purpose of assessment and evaluation is to support student learning" (p. 6). Methods of assessment and evaluation need to be reconsidered in light of the goals of the destreamed course. How can assessment and evaluation include allowing multiple ways for students to demonstrate their learning and understanding? How are we to find ways to do such assessments for a destreamed range of student understanding? What options are there to assess *more* than skills and procedures?

It is critically important that assessment and evaluation methods echo what we are professing to *value* in learning, and whether the idea of assessing thinking is to be included. If we truly want students to think, to show their understanding in multiple ways, and to solve relevant, real-world problems that have meaning for them, then we must *value* that in our assessment and "e-*value*-ation."

I have, for a long time, been an advocate of tasks as a viable evaluation method (Atiya et al., 2018) and have used this method with Grades 9 and 10 students under the previous curriculum, with positive results (Kajander, 2002). The removal of the stress of a time limit, as well as the opportunity this creates for students to actually *do* problem solving, allows for a much richer evaluation.

I have found the use of a learning task/evaluation task pair to be especially helpful in clarifying to students what is expected of them, and providing them with opportunities to improve (OME, 2010). For students who have had less experience with task-based evaluation of problem-based learning, it has a different feel, and they can initially find it confusing. This article will provide some background, and then provide a sample pair of tasks, ready for classroom use.

Learning Tasks and Evaluation Tasks

In this article, the benefit of having two parallel tasks, which share curriculum Expectations, but involve a different context or a different aspect of the same context, will be explored. The first task, which we refer to as a "Learning Task," may be done with students working alone or in pairs or small groups. The teacher may give substantive feedback in the moment, and also provide detailed feedback on a required written product. It is not graded, but rather, provides the student(s) the opportunity to experience working on a problem-solving task, using a given set of curriculum Expectations (including the process Expectations), with teacher support and feedback for the purpose of improving. Students can work with peers, ask the teacher questions, and get substantive, constructive feedback, to subsequently improve their understanding and demonstrate this on the subsequent task, which we will call the Evaluation Task. As well, as students are working on the Learning Task, teachers can be observing the mathematical processes, in the moment. The Evaluation Task can also use a subset of the Learning Task Expectations, but shouldn't attempt to evaluate more of them than were addressed in the Learning Task. The mathematical processes (Strand A: Representing, Reasoning, Problem-Solving, etc) can be front and centre in these tasks, and remembering the other details of Strand A, socio–emotional learning goals, is also important. Engagement is enhanced when students see the tasks as relevant and interesting, and they are empowered to see the value and use of the mathematics they have learned in addressing real problems in their lives. While this strand does not need to be graded, it should still be a consideration in task design and feedback to students.

The use of a task-specific rubric allows students to demonstrate their understanding in various ways, and at a range of levels. If the tasks are rich enough, then extended research or high-ceiling opportunities are helpful to keep high-performing students engaged and learning, and rubrics can also reflect this. The rubric should be given to students with the Evaluation Task, to ensure evaluation is transparent. Afterwards, once the tasks have been used in a given semester, exemplars of student work can be kept and provided to future students to illustrate examples of levels of performance.

Practical Considerations

The reality of many classrooms is that students work at different paces and are sometimes absent. For this reason, in my own classroom practice, I found it helpful to have the two tasks (Learning Task and Evaluation Task) printed on different colours of paper. That way, I could see at a glance who was working on what. Giving consideration to the timing, so that the teacher can provide the feedback in a timely manner before the student moves on to the Evaluation Task without too much delay, is beneficial. Another option is to have an in-between day, where students are simply building skills and fluency by practising techniques with textbook problems. When some students are behind in the required classroom work, teachers have to make a judgment call about what to prioritize and what to leave off; it may be the case that at times, some of this skills practice is omitted for students working more slowly. Such decisions will be influenced by one's values and priorities; the general idea here is that *understanding* precedes fluency.

A pair of sample tasks is provided for use with the Grade 9 destreamed course. They were designed by a teacher candidate, in a curriculum and instruction course in secondary mathematics, as part of our course's culminating activity. We would be delighted to receive feedback on the tasks from anyone who tries them out, and welcome suggestions for improvement. The Evaluation Task that follows is situated geographically in Thunder Bay—but of course, it can be adjusted to any region.

Sample Learning and Evaluation Tasks – Strands B., C., and E.

MTH1W – Grade 9 Mathematics

Summary of Strands: B. Number, C. Algebra, E. Geometry and Measurement

Theme: Climate Change

Amount of Time Per Task: 70 minutes (Learning Task), 210 minutes (Evaluation Task)

Expectations in bold type in the following list are found in both the Learning and Evaluation Tasks. Expectations not highlighted are only found in the Learning Task.

Throughout this course, in connection with the learning in the other strands, students will:

A2 Making Connections
make connections between mathematics and various knowledge systems, their lived experiences, and various real-life applications of mathematics, including careers;

B3 Number Sense and Operations
apply an understanding of rational numbers, ratios, rates, percentages, and proportions, in various mathematical contexts, and to solve problems;

B3.5 pose and solve problems involving rates, percentages, and proportions in various contexts, including contexts connected to real-life applications of data, measurement, geometry, linear relations, and financial literacy;

C1 Algebraic Expressions and Equations;

C1.2 create algebraic expressions to generalize relationships expressed in words, numbers, and visual representations, in various contexts;

C1.3 compare algebraic expressions, using concrete, numerical, graphical, and algebraic methods to identify those that are equivalent, and justify their choices;

C1.5 create and solve equations for various contexts, and verify their solutions;

C3 Application of Relations
represent and compare linear and non-linear relations that model real-life situations, and use these representations to make predictions;

C3.3 compare two linear relations of the form $y = ax + b$ graphically and algebraically, and interpret the meaning of their point of intersection in terms of a given context;

E1.3 solve problems involving different units within a measurement system and between measurement systems, including those from various cultures or communities, using various representations and technology, when appropriate;

E1.5 solve problems involving the side-length relationship for right triangles in real-life situations, including problems that involve composite shapes.

Grade 9 Learning Task

How Green Are Electric Vehicles?

Although Carbon Dioxide (CO_2) is not emitted when electric motors are running, the electricity used by Electric Vehicles (EVs) may generate CO_2 during their production, and the amounts depend on which sources of energy from which the electricity originates. The more fossil fuels are used to generate electricity, the more CO_2 will result.

1) A mid-sized Tesla Model 3 electric vehicle generates 8100 kilograms of CO_2 to manufacture. While electric motors do not emit CO_2 when they run, CO_2 can be emitted during the generation of electricity that EVs draw from the electrical grid and store in their batteries. In the United States, 23% of electricity comes from coal-fired plants. From this electricity source, the Tesla Model 3 will still emit 72.7g of CO_2 per kilometer. A similar gasoline vehicle, the Toyota Corolla, generates 5500 kilograms of CO_2 to manufacture, and emits 192.4 of CO_2 per kilometer.
   a) Which vehicle results in more CO_2 during manufacturing? How much more?
   b) Assuming both vehicles have an average lifespan of 280 000 kilometres, how much CO_2 does each vehicle produce during its lifetime, including manufacturing?
   c) How many kilometers does the Tesla 3 Model have to drive before it becomes cleaner than a Toyota Corolla in terms of its CO_2 emissions?
   d) The Tesla Model 3 emits 72.7g of CO_2 per kilometer on a grid where 23% of its electricity comes from coal. How much would it emit if only 10% of its electricity came from coal (and assuming there are no emissions from any other source on the grid)?
2) The carbon intensity of electricity is the rate at which CO_2 is released per unit of electricity generated, and can be expressed in grams of CO_2 per kilowatt hours (g CO_2/kWh) per kilometer. The map above displays the carbon intensity of each province.
   a) Which provinces have the lowest CO_2 emitting energy in Canada?
   b) A Tesla Model X is travelling from point A in Saskatchewan to point B in Nunavut. The car first travels 300 km east to point C in Manitoba, using the Saskatchewan power grid, and emits 163.3g CO_2/km. At point C it recharges its batteries and emits 0.5g /km as it travels an additional 400 km north to point B. Using the Saskatchewan power grid, the car emits 163.3g CO_2/km during the entire trip. How would this change if the car is recharged in Manitoba?
   c) If the car had travelled from point A to point B in a straight line, using the Saskatchewan power grid, how much more CO_2 would have been emitted?

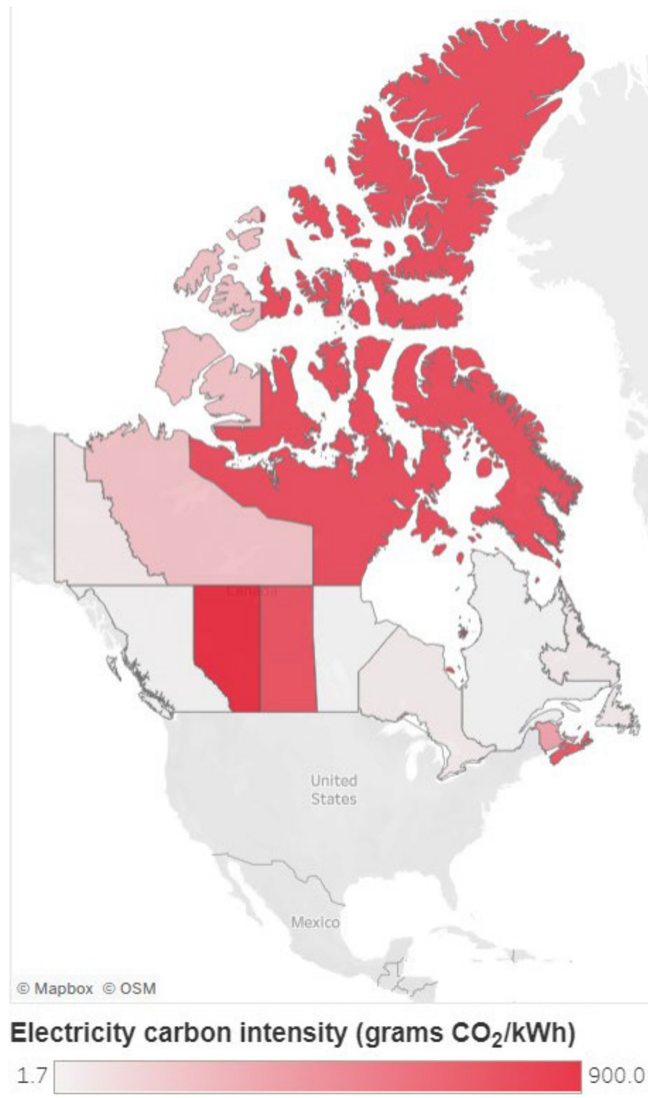


Figure 1: *Carbon intensity of electricity production*

Grade 9 Assessment Task

The total amount of greenhouse gases, such as carbon dioxide (CO_2), produced by human actions is known as a carbon footprint.

One of the highest rates in the world, the average carbon footprint of a person in Canada is 14.2 tonnes per year. The average carbon footprint across the globe is closer to 4 tonnes per year. By 2050, the average worldwide carbon footprint must fall to under 2 tonnes per year to have the best chance of preventing a rise in global temperatures of 2°C.

Individual carbon footprint reduction from 16 tonnes to 2 tonnes takes time! We can start to have a significant impact by making minor adjustments to our behaviour, such as eating less meat, booking fewer connecting flights, and line-drying our clothes.

Task

Stevan lives in Thunder Bay. He wants a ride from Westfort in Fort William to Chapples Park. He can take an Uber, a Lyft, or use public transportation. When checking the routes on his phone, shown in Figure 2, he finds the CO_2 emissions per kilometre of each ride. Additionally, Stevan finds that the Lyft would emit another 285 grams of CO_2 on the way to pick him up.

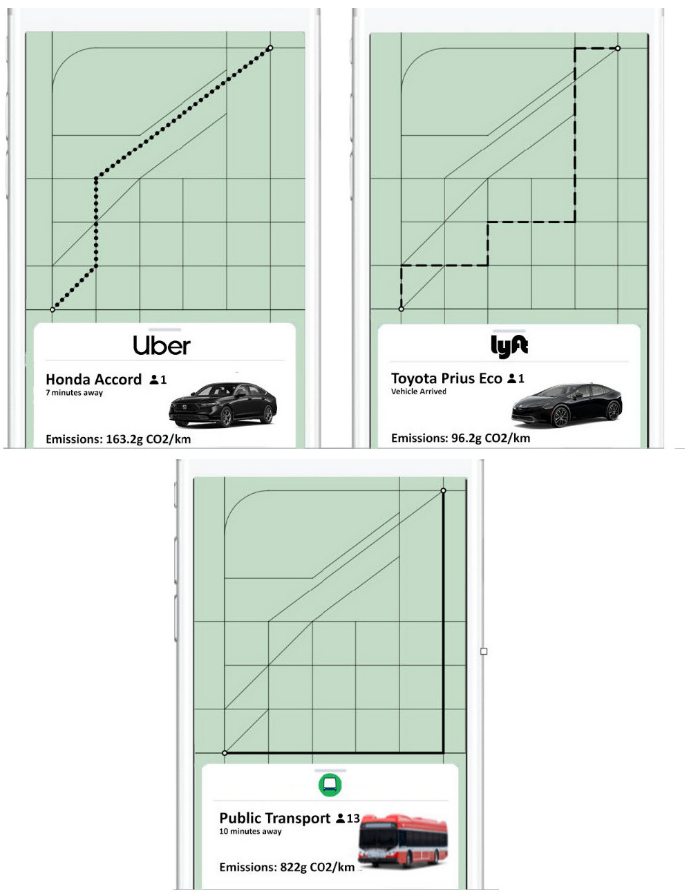


Figure 2: *Ride options*

1. What is the total amount of CO_2 that would be emitted by each ride?
2. At what point does the Uber's emissions exceed the Lyft's CO_2 emissions?
3. Which ride will result in the lowest amount of emissions per passenger?
4. Which ride should Stevan take to minimize his carbon footprint? Explain how you know.

To assist with choosing which ride Stevan should take, he laid out all three routes on a grid. Each unit, or two centimeters, on the grid represent 1 kilometer.

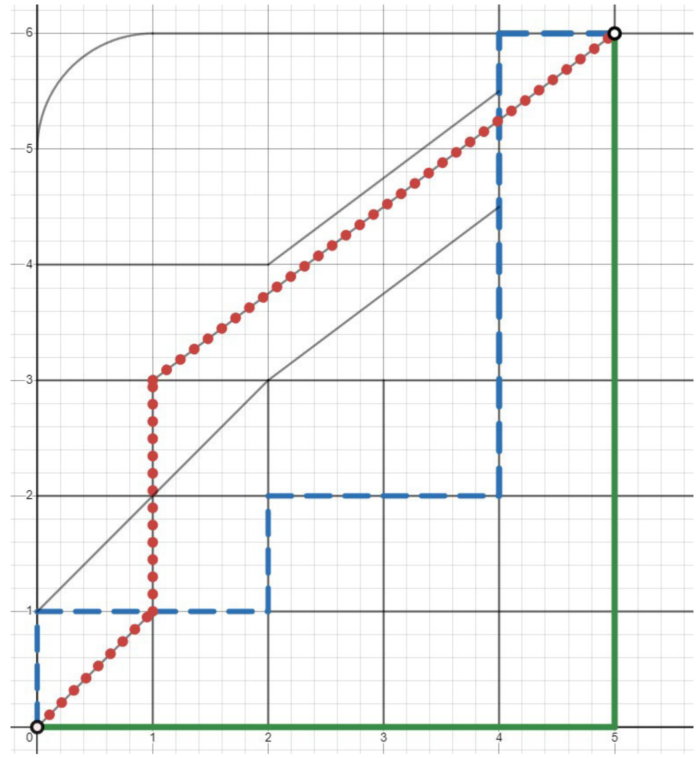


Figure 3: *Different routes*

Presentation

Your final product should include the following information:

- A thorough explanation of the steps you took to solve the problem
- The calculations you made, presented in a clear and concise manner
- All calculations rounded to 2 decimal places
- If the length of the routes on the grid were measured using a ruler, clearly indicate which part was measured, its length, and the calculations you used to convert it to kilometers.
- A statement indicating when the Uber's emissions exceed the Lyft's emissions
- A conclusion that states the CO_2 emission of each ride, and the ride that results in the lowest CO_2 emissions per person, including the ride that would minimize Stevan's carbon footprint, along with an explanation of which ride Stevan should take to minimize his carbon footprint

RUBRIC				
Student Name: ______________________				
Expectations	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Knowledge / Understanding	• Demonstrates limited understanding of the relationship between CO2 emissions per kilometer and distance travelled.	• Demonstrates some understanding of the relationship between CO2 emissions per kilometer and distance travelled.	• Demonstrates considerable understanding of the relationship between CO2 emissions per kilometer and distance travelled.	• Demonstrates thorough understanding of the relationship between CO2 emissions per kilometer and distance travelled.
Thinking	• Routes are split up to calculate distance in a limited degree. • Statements and conclusion are formed with limited degree of accuracy.	• Routes are split up in a somewhat logical fashion to calculate distance. • Statements and conclusion are formed with some degree of accuracy.	• Routes are split up in a generally logical fashion to calculate distance. • Statements and conclusion are formed with a considerable degree of accuracy.	• Routes are split up in a highly logical fashion to calculate distance. • Statements and conclusion are formed with a high degree of accuracy.
Communication	• Expresses and organizes ideas and information with limited effectiveness. • Uses mathematical language, symbols, and units with limited effectiveness.	• Expresses and organizes ideas and information with some effectiveness. • Uses mathematical language, symbols, and units with some effectiveness.	• Expresses and organizes ideas and information with considerable effectiveness. • Uses mathematical language, symbols, and units with considerable effectiveness.	• Expresses and organizes ideas and information with a high degree of effectiveness. • Uses mathematical language, symbols, and units with a high degree of effectiveness.
Application	• Demonstrates limited skill in the application of the formula.	• Demonstrates some amount of skill in the application of the formula.	• Demonstrates a considerable amount of skill in the application of the formula.	• Demonstrates a high degree of skill in the application of the formula.

Resources and Sources Used in Task Design

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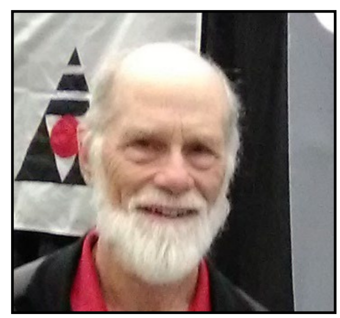
▲ Ontario Mathematics Advertising Manager — Robert Sherk



Since becoming the Editor, I have tried to include some recognition of the members of the *Gazette* team. However, catching up with them is not easy because none of them stand still for very long. In the case of Robert Sherk, I was well aware his engagement with OAME goes back generations. There is a picture in the *Gazette* of three generations of the Sherk family at an OAME event.

Robert has tirelessly, and without fanfare, managed advertising for the *Gazette* since June 2007, when he took the reins from Dean Murray. Since then, he has looked after it to the extent that, as Editor, I focus on other things, knowing Robert will do what he does so well.

Thank you, Robert!

FROM THE GALLERY: A MATHEMATICS-TEACHING FAMILY — PAUL SHERK, JOHN STOUFFER (MATH HEAD), BETTY SHERK (GRADE TWO TEACHER), JOHN SHERK (MATH TEACHER), AND BOB SHERK (MATH & PHYSICS TEACHER) — OAME ANNUAL CONFERENCE, 1980

Photo: *From 1991,* Ontario Mathematics Gazette, *29(3)*

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