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ONTARIO MATHEMATICS GAZETTE

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EDITORIAL

As I write this editorial, it is very appropriate to wish all readers of the Gazette a very Happy New Year in 1964. By the time you read the Editorial, I admit that the greeting will be somewhat belated.

The first thing which I would like to bring up this year is a grievance which I encounter every Fall when registering freshman students. This is connected with students who do not take three Mathematics papers in Grade XIII but take only one paper or two papers. These students should be receiving better advice from their mathematics teachers, their principals, or their guidance teachers. Two difficult situations keep arising. Last Fall I encountered one student who had First Class Honours in eight papers (two English, two French, two Latin, two Biology) and had failed Grade XIII Geometry miserably. Now this student liked History and would have done well in History; yet she was advised to take Geometry as her ninth paper for University entrance. This situation should not arise. I realize that there will be students who use one or two mathematics papers to fill up their nine papers for University entrance; however, individual mathematics papers should only be used as such fillers in the very last resort. A single mathematics paper is of very little use to the student except as filler.

A much more serious situation arises when a student enters University with only two mathematics papers. I am always meeting students who are quite perplexed and say "But no one ever told me", when I point out to them that, in order to take First Year University Mathematics, they should have standing in the three Grade XIII Mathematics papers. In fact, any student with fewer than three papers is not going to pass First Year Calculus unless he is a student of such outstanding calibre that he can get up the missing paper on his own. I grant that very few students coming in with two Mathematics papers wish to specialize in Mathematics at University (surprisingly enough, there are always some). But many students who are going into Biology or into the Social Sciences do enter University with insufficient background in Mathematics. This is due to an insufficient realization on the part of advisers of what modern developments in the Social and Biological Sciences have been.

It is perfectly true that, a number of years ago, when a subject like Psychology was in the hand-holding and unscientific stage, it was not necessary to know very much about Mathematics. All this has changed. The Biological and Social Sciences have become true sciences and are developing along quantitative lines. Any modern and respectable Biology course will involve a minimum of a course in Calculus, a course in Differential Equations, and a course in Probability and Statistics. These are essential for the student to be able to follow modern work in population studies and modern work in mathematical genetics. The day of the "Statistics" course which did not involve a knowledge of Calculus has completely passed away (there never really was a place for such a course in a proper treatment).

The change in the Social Sciences has been even more striking.

Proper work in modern Psychology involves not only a knowledge of Calculus, Differential Equations, Probability and Statistics, but involves further work in the field of Probability and Statistics so that the student can understand what is involved in the proper design of experiments, the amount of information retrievable from an experiment, and the principles of statistical inference. Indeed, the distinction between good and bad research in modern psychology usually depends upon the amount of knowledge about the logic of statistical inference possessed by the experimenter. There never was a time when psychology was properly a branch of "Pastoral Theology"; but nowadays there are very few people left who even think that it is.

My conclusion from the foregoing remarks is the following. Do not tell students that if they are going into Economics or if they are going into Psychology, they can get along without mathematics. They need just as much high school mathematics as do students who are going into Mathematics, Physics, or Chemistry. And they will need to obtain much more mathematics when they get to university.

And now to change the subject for a while. You have probably noticed that there is now a change on our front cover. The Samuel Beatty Fund has been added to the list of sponsors of the Ontario Mathematics Gazette. As you are aware, the Gazette was founded and supported by the Ontario Mathematics Commission and the Ontario Association of Teachers of Mathematics and Physics. We are very grateful for their encouragement and financial assistance. The Board of Trustees of the Samuel Beatty Fund has now given us a grant to assist in the publication of the Gazette for the current year, and we are naturally hopeful that they will do the same in the future. This grant will allow us to publish four issues of the Gazette per year (the usual 36-page issues) rather than the three issues which we have been publishing in the past. However, in the current year (1964), our planning for this change was not able to be made quite as early as we had anticipated, and consequently, for reasons of convenience, rather than publish four issues of the Gazette, we will be publishing only three issues. These issues will be of forty-eight pages each and so will be the equivalent in content of four issues. This increase in the size, and we hope the usefulness, of the Ontario Mathematics Gazette is due to the generosity of the Samuel Beatty Fund, and we are extremely grateful for the grant which they have made.

One final word - we are receiving a good number of very worthwhile articles contributed to the Gazette. Keep up the good work; the more articles that are submitted, the better selection we will have.

Ralph G. Stanton

Editor-in-Chief

[Editor's Note: The following article has been contributed by one of Canada's leading mathematicians as an indication of the level of knowledge about exponents which he believes is desirable.]

E X P O N E N T S

The following notes are mainly intended to collect the information on exponents which the authors believe may be useful to the teacher. They emphatically do not advocate an introductory treatment of exponents along the lines of these notes.

Notation: Greek letters denote real numbers. Unless stated differently they are assumed to be different from zero.

The letters m, n, p, q, denote integers.

1. NON-NEGATIVE INTEGRAL EXPONENTS

The following definition has been chosen because it seems both economical and instructive.

Definition 1: Let α be any real number, $\alpha^0 = 1$. If α^n has been defined for the non-negative integer n, define

$$\alpha^{n+1} = \alpha^n.\alpha.$$

Remark 1: By this definition, $\alpha^1 = \alpha^{0+1} = \alpha^0.\alpha = 1.\alpha = \alpha$, $\alpha^2 = \alpha^1.\alpha = \alpha.\alpha$, etc. A definition of this type is called a "definition by induction".

Theorem 1: $\alpha^n \neq 0$ for all $n \geq 0$.

Proof: By Definition 1, $\alpha^0 \neq 0$; by Remark 1, $\alpha^1 \neq 0$. Suppose we have already proved that Theorem 1 holds true for all $n \leq m$. Then $\alpha^{m+1} = \alpha^m.\alpha$ is the product of two non-vanishing factors. Hence $\alpha^{m+1} \neq 0$. Thus Theorem 1 is true for $n = m + 1$. This proves Theorem 1 by induction.

Theorem 2: $1^n = 1$ for all $n \geq 0$.

Proof by induction: By Definition 1 and Remark 1, Theorem 2 is true for $n = 0$ and $n = 1$. Suppose it has been proved for $n \leq p$. Then by Definition 1 and our induction assumption

$$1^{p+1} = 1^p.1 = 1.1 = 1.$$

This proves Theorem 2 for $n = p + 1$.

Theorem 3: $\alpha^n.\alpha^m = \alpha^{n+m}$ for all $n \geq 0$, $m \geq 0$.

Proof: We keep n fixed. By Definition 1 and Remark 1, we have

$$\alpha^n.\alpha^0 = \alpha^n.1 = \alpha^n = \alpha^{n+0} \quad \text{and} \quad \alpha^n.\alpha^1 = \alpha^n.\alpha = \alpha^{n+1}.$$

Thus Theorem 3 holds true for $m = 0$ and $m = 1$. Suppose it has been proved for all $m \leq p$. Then by Definition 1 and our induction assumption

$$\alpha^n.\alpha^{p+1} = \alpha^n(\alpha^p.\alpha) = (\alpha^n.\alpha^p).\alpha = \alpha^{n+p}.\alpha = \alpha^{(n+p)+1} = \alpha^{n+(p+1)}.$$

This proves Theorem 3 for $m = p + 1$ and completes the induction step.

Theorem 4: $\alpha^n \div \alpha^m = \alpha^{n-m}$ if $n \geq m \geq 0$.

Proof: Since $n - m \geq 0$ we may apply Theorem 3 with $n - m$ instead of n obtaining

$$\alpha^{n-m}.\alpha^m = \alpha^{(n-m)+m} = \alpha^n.$$

On account of Theorem 1, we may divide by α^m. This yields our statement.

Theorem 5: $(\alpha^n)^m = \alpha^{nm}$ if $n \geq 0$, $m \geq 0$.

Proof: Keep n fixed. By Theorem 1, $\alpha^n \neq 0$. Hence we may apply Definition 1 and Remark 1 obtaining

$$(\alpha^n)^0 = 1 = \alpha^0 = \alpha^{n.0} \quad \text{and} \quad (\alpha^n)^1 = \alpha^n = \alpha^{n.1}.$$

This proves the cases $m = 0$ and $m = 1$ of Theorem 1. Suppose this theorem has been proved for $m \leq p$. Then by Definition 1 and Theorem 3,

$$(\alpha^n)^{p+1} = (\alpha^n)^p.\alpha^n = \alpha^{np}.\alpha^n = \alpha^{np+n} = \alpha^{n(p+1)}.$$

Thus Theorem 5 remains true if $m = p + 1$. This proves our theorem by induction.

Theorem 6: $\alpha^n.\beta^n = (\alpha\beta)^n$ if $n \geq 0$.

Proof: Since $\alpha \neq 0$, $\beta \neq 0$, we have $\alpha\beta \neq 0$. Thus by Definition 1 and Remark 1, we have

$$\alpha^0.\beta^0 = 1.1 = 1 = (\alpha\beta)^0 \quad \text{and} \quad \alpha^1.\beta^1 = \alpha\beta = (\alpha\beta)^1.$$

This proves Theorem 6 for $n = 0$ and $n = 1$. If it is proved for $n \leq p$, Definition 1 and our induction assumption yield

$$a^{p+1}.\beta^{p+1} = (\alpha^p.\alpha)(\beta^p.\beta) = (\alpha^p.\beta^p).(\alpha\beta) = (\alpha\beta)^p.(\alpha\beta) = (\alpha\beta)^{p+1}.$$

Thus Theorem 6 remains valid for $n = p + 1$.

Theorem 7: $\alpha^n \div \beta^n = (\alpha/\beta)^n$ if $n \geq 0$.

Proof: Since $\alpha \neq 0$, $\beta \neq 0$, the quotient α/β exists and is not zero. Apply Theorem 6, replacing α by α/β. This yields

$$(\alpha/\beta)^n.\beta^n = [(\alpha/\beta)\beta]^n = \alpha^n.$$

By Theorem 1, $\beta^n \neq 0$. Thus we may divide by β^n obtaining our Theorem.

Remark 2: We have been careful to exclude powers of zero from the preceding discussion. It is customary to define

$$0^n = 0 \quad \text{if} \quad n > 0.$$

With this definition, Theorems 3 and 5 remain valid if $\alpha = 0$, $n > 0$, and $m > 0$. Theorem 6 remains correct for $n > 0$ if α or β or both vanish.

[Occasionally it is convenient either to define $0^0 = 0$ or to define $0^0 = 1$. Either definition enables us to allow $\alpha = 0$ or $\beta = 0$ in Theorems 3, 5, and 6. However, both definitions cause

difficulties in calculus.]

2. INTEGRAL EXPONENTS

Definition 2: $\alpha^{-n} = 1/\alpha^n$ if $n > 0$.

We note that this definition is permissible on account of Theorem 1.

Theorem 8: $\alpha^{-n} = (1/\alpha)^n$ for all n.

Proof: If $n > 0$, Definition 2 and Theorems 2 and 7 yield

$$\alpha^{-n} = 1/\alpha^n = 1^n/\alpha^n = (1/\alpha)^n.$$

If $n = 0$, we obtain from Definition 1

$$\alpha^{-0} = \alpha^0 = 1 = (1/\alpha)^0.$$

Let $n = -p < 0$. We apply the first part of our proof replacing n by p and α by $1/\alpha$. Thus

$$(1/\alpha)^{-p} = (1/(1/\alpha))^p = \alpha^p.$$

But the right hand term is equal to α^{-n} and the left one to $(1/\alpha)^n$.

Remark 3: We could have replaced Definition 2 by the

Definition: $\alpha^{-n} = (1/\alpha)^n$ if $n > 0$

and proved the

Theorem: $\alpha^{-n} = 1/\alpha^n$ if $n > 0$.

Theorem 1': $\alpha^n \neq 0$ for all n.

Proof: On account of Theorem 1 we may assume $n < 0$. Let $m = -n > 0$. By Definition 2

$$\alpha^n = \alpha^{-m} = 1/\alpha^m \neq 0.$$

Theorem 2': $1^n = 1$ for all n.

Proof: If $n \geq 0$, apply Theorem 2. Let $n = -m < 0$. By Theorems 8 and 2

$$1^n = 1^{-m} = (1/1)^m = 1^m = 1.$$

Theorem 3': $\alpha^n.\alpha^m = \alpha^{n+m}$ for all n,m.

Proof: We distinguish several cases.

Case (i): $n \geq 0$, $m \geq 0$. In this case, apply Theorem 3.

Case (ii): $n \geq 0$, $m < 0$, $n + m \geq 0$. Put $p = -m$. Thus

$$n \geq -m = p > 0$$

and by Definition 2 and Theorem 4

$$\alpha^n.\alpha^m = \alpha^n.\alpha^{-p} = \alpha^n.\frac{1}{\alpha^p} = \alpha^{n-p} = \alpha^{n+m}.$$

Case (iii): $n \geq 0$, $m = -p < 0$, $n + m < 0$. In this case $p = -m > n \geq 0$. By Definition 2 and Theorem 4

$$\alpha^n.\alpha^m = \frac{\alpha^n}{\alpha^p} = 1/\frac{\alpha^p}{\alpha^n} = 1/\alpha^{p-n} = \alpha^{-(p-n)} = \alpha^{n-p} = \alpha^{n+m}.$$

By the preceding discussion, Theorem 3' is valid if $n \geq 0$. Symmetrically it will be true for $m \geq 0$. Thus it remains to deal with

Case (iv): $n < 0$, $m < 0$. Put $n = -p$, $m = -q$. Then by Definition 2 and Theorem 3

$$\alpha^n.\alpha^m = \alpha^{-p}.\alpha^{-q} = \frac{1}{\alpha^p} \cdot \frac{1}{\alpha^q} = \frac{1}{\alpha^p.\alpha^q} = \frac{1}{\alpha^{p+q}} = \alpha^{-(p+q)}$$

$$= \alpha^{-p-q} = \alpha^{n+m}.$$

This completes the proof of Theorem 3'.

Theorem 4': $\alpha^n \div \alpha^m = \alpha^{n-m}$ for all n, m.

Proof: By Theorem 3'

$$\alpha^m.\alpha^{n-m} = \alpha^{m+(n-m)} = \alpha^n.$$

By Theorem 1' we may divide both sides by α^m.

Theorem 5': $(\alpha^n)^m = \alpha^{nm}$ for all n, m.

Proof: If $n \geq 0$, $m \geq 0$, we may apply Theorem 5. Consider now the case $n \geq 0$, $m < 0$. Put $p = -m$. Then by Definition 2 and Theorem 5

$$(\alpha^n)^m = (\alpha^n)^{-p} = \frac{1}{(\alpha^n)^p} = \frac{1}{\alpha^{np}} = \alpha^{-np} = \alpha^{nm}.$$

This proves Theorem 5' if $n \geq 0$.

Let $n = -q < 0$. Then Theorem 8 and the first part of our proof imply

$$(\alpha^n)^m = (\alpha^{-q})^m = [(\frac{1}{\alpha})^q]^m = (\frac{1}{\alpha})^{qm} = \alpha^{-qm} = \alpha^{nm}.$$

Theorem 6': $\alpha^n.\beta^n = (\alpha\beta)^n$ for all n.

Proof: On account of Theorem 6, we may assume $n = -p < 0$. By Definition 2 and Theorem 6

$$\alpha^n.\beta^n = \alpha^{-p}.\beta^{-p} = \frac{1}{\alpha^p} \cdot \frac{1}{\beta^p} = \frac{1}{\alpha^p.\beta^p} = \frac{1}{(\alpha\beta)^p} = (\alpha\beta)^{-p} = (\alpha\beta)^n.$$

Theorem 7': $\alpha^n/\beta^n = (\alpha/\beta)^n$ for all n.

Proof: By Theorem 6'

$$\beta^n.(\alpha/\beta)^n = (\beta.\alpha/\beta)^n = \alpha^n.$$

3. RATIONAL EXPONENTS

Notation: From now on, Greek letters denote positive real numbers.
The letters m, n, p, q again denote integers;
r and s are rational numbers.

Theorem 9: $\alpha^n > 0$ for all n.

Proof: By Definition 1 and Remark 1, this is true for $n = 0$ and $n = 1$. Suppose it has been proved for

$$n = 0, 1, \ldots, p, \qquad [\text{thus } p > 0].$$

Then $\alpha^{p+1} = \alpha^p.\alpha$ is the product of two positive real numbers and hence positive. This proves our theorem for all positive n's.

Let $n = -m$ be negative. By the first part of our proof, $\alpha^n = 1/\alpha^m$ is the reciprocal of a positive number and therefore also positive. This completes our proof.

Theorem 9 is needed for the proof of the following important lemma:

Theorem 10: Let $n \neq 0$, $\alpha^n = 1$. Then $\alpha = 1$.

Proof: Case (i): $n > 0$. Let $\alpha^n = 1$. Then

$$0 = \alpha^n - 1 = (\alpha - 1)(1 + \alpha + \ldots + \alpha^{n-1}). \tag{1}$$

By Theorem 9,

$$1 + \alpha + \ldots + \alpha^{n-1} \geq 1 + 0 + \ldots + 0 = 1,$$

in particular

$$1 + \alpha + \ldots + \alpha^{n-1} \neq 0.$$

Hence (1) implies $\alpha = 1$.

Case (ii): $n = -p > 0$. Let $\alpha^n = 1$. Then by Definition 2

$$1 = \alpha^{-p} = 1/\alpha^p \quad \text{or} \quad \alpha^p = 1.$$

By Case (i), this implies $\alpha = 1$.

Theorem 11: The equation

$$\xi^n = \alpha^m \tag{2}$$

always has one and only one solution ξ for given α, m, and n $(n \neq 0)$.

Proof: A rigorous proof of the existence of a solution requires calculus. However, the uniqueness can readily be deduced from Theorem 10.

Suppose

$$\xi^n = \alpha^m \quad \text{and} \quad \eta^n = \alpha^m.$$

Then $\xi^n = \eta^n$ and by Theorem 7'

$$(\xi/\eta)^n = \xi^n/\eta^n = 1.$$

Since $n \neq 0$, Theorem 10 implies

$$\xi/\eta = 1 \quad \text{or} \quad \xi = \eta.$$

Definition 3: The solution ξ of (2) is denoted by

$$\xi = \alpha^{\frac{m}{n}}.$$

This definition must be justified by the following theorem.

Theorem 12: Let $m/n = p/q$ [thus $n \neq 0$ and $q \neq 0$]. Then

$$\alpha^{m/n} = \alpha^{p/q}.$$

Proof: Put

$$\xi = \alpha^{m/n}, \quad \eta = \alpha^{p/q}.$$

Thus by Definition 3

$$\xi^n = \alpha^m, \quad \eta^q = \alpha^p.$$

Since $mq = pn$, this implies by Theorem 5'

$$\xi^{nq} = (\xi^n)^q = (\alpha^m)^q = \alpha^{mq} = \alpha^{pn} = (\alpha^p)^n = (\eta^q)^n = \eta^{qn}.$$

Hence by Theorem 7'

$$(\xi/\eta)^{nq} = \xi^{nq}/\eta^{nq} = 1.$$

Since $nq \neq 0$, Theorem 10 yields

$$\xi/\eta = 1 \quad \text{or} \quad \xi = \eta.$$

Theorem 2": $1^r = 1$ for all r.

Proof: Let $r = m/n$. Thus $n \neq 0$. Put $\xi = 1^{m/n}$. Then by Definition 3 and Theorem 2'

$$\xi^n = 1^m = 1.$$

Hence $\xi = 1$, by Theorem 10.

Theorem 3": $\alpha^r.\alpha^s = \alpha^{r+s}$ for all r, s.

Proof: Let $r = m/n$, $s = p/q$. Thus $nq \neq 0$ and $r + s = \frac{mq+np}{nq}$.

Put

$$\xi = \alpha^r, \quad \eta = \alpha^s. \quad \text{Thus} \quad \xi^n = \alpha^m, \quad \eta^q = \alpha^p.$$

We have to show

$$\xi\eta = \alpha^{r+s} \quad \text{or} \quad (\xi\eta)^{nq} = \alpha^{mq+np}.$$

But by Theorems 6', 5', and 3'

$$(\xi\eta)^{nq} = \xi^{nq}.\eta^{nq} = (\xi^n)^q.(\eta^q)^n = (\alpha^m)^q.(\alpha^p)^n$$

$$= \alpha^{mq}.\alpha^{pn} = \alpha^{mq+np}.$$

Theorem 4": $\alpha^r/\alpha^s = \alpha^{r-s}$ for al r, s.

Proof: By Theorem 3"

$$\alpha^s.\alpha^{r-s} = \alpha^{s+(r-s)} = \alpha^r.$$

But $\alpha^s > 0$ by Theorem 10 and Definition 3.

Theorem 5": $(\alpha^r)^s = \alpha^{rs}$ for all r,s.

Let $r = m/n$, $s = p/q$. Put

$$\xi = \alpha^r, \quad \eta = \xi^s = (\alpha^r)^s. \quad \text{Thus} \quad \xi^n = \alpha^m, \quad \eta^q = \xi^p$$

and

$$\eta^{qn} = (\eta^q)^n = (\xi^p)^n = \xi^{pn} = (\xi^n)^p = (\alpha^m)^p = \alpha^{mp}.$$

Hence

$$\eta = \alpha^{\frac{mp}{qn}} = \alpha^{rs}.$$

Theorem 6": $\alpha^r.\beta^r = (\alpha\beta)^r$ for all r.

Proof: Let $r = m/n$, $\xi = \alpha^r$, $\eta = \beta^r$. Thus $\xi^n = \alpha^m$, $\eta^n = \beta^m$ and by Theorem 6'

$$(\xi\eta)^n = \xi^n\eta^n = \alpha^m\beta^m = (\alpha\beta)^m.$$

Hence by Definition 3

$$\xi\eta = (\alpha\beta)^{m/n} = (\alpha\beta)^r.$$

Theorem 7": $\alpha^r/\beta^r = (\alpha/\beta)^r$ for all r.

Proof: $\beta^r.(\alpha/\beta)^r = (\beta.\alpha/\beta)^r = \alpha^r$ by Theorem 6".

Remark 4: The introduction of rational powers of negative real numbers is not recommended [cf. however Remark 5]. The equation (2) may have two or no real solutions and an indiscriminate application of the preceding theorems may lead to absurdities.

Example: $-1 = (-1)^1 = (-1)^{2/2} = [(-1)^2]^{1/2} = 1^{1/2} = ?$

Remark 5: The theory of equations requires the introduction of surds not only of negative but of arbitrary complex numbers. The discussion of these surds should be carefully kept separate from the preceding theory of exponents. A unified theory requires complex analysis.

Solutions to Grade 13 Problems, 1963

1. The ratio of male to female votors at an election was a:b. If c fewer men and d fewer women had voted, the ratio would have been p:q. Prove that the total number of votes cast was

$$(a + b)(qc - pd)/(qa - pb).$$

Solution: Let M denote the number of male voters, and F the number of female voters. The given information implies that

$$\frac{M}{F} = \frac{a}{b}, \qquad \frac{M-c}{F-d} = \frac{p}{q}.$$

Solving these two linear equations for M, F, yields

$$M = a(qc - pd)/(qa - pb),$$

$$F = b(qc - pd)/(qa - pb);$$

hence, the total number of voters is given by

$$M + F = (a + b)(qc - pd)/(qa - pb).$$

2. Suppose a, b, c, and d are positive numbers. Prove that

(i) $a^4 + b^4 \geq 2a^2b^2$,

(ii) $a^4 + b^4 + c^4 + d^4 \geq 4\,abcd$,

(iii) $(a^2 + b^2)^2 + (c^2 + d^2)^2 \geq 2(ab + cd)^2$.

Solution: For any x, y, $(x-y)^2 \geq 0$ or $x^2 + y^2 \geq 2xy$. The following inequalities are special cases of this inequality.

(1) $(a^2)^2 + (b^2)^2 > 2a^2b^2$

(2) $(c^2)^2 + (d^2)^2 \geq 2c^2d^2$

(3) $(ab)^2 + (cd)^2 \geq 2abcd$

(4) $a^2 + b^2 \geq 2ab$

(5) $c^2 + d^2 \geq 2cd$.

The inequality (i) is (1). Now using (1), (2), and (3),

$$a^4 + b^4 + c^4 + d^4 \geq 2a^2b^2 + 2c^2d^2 \geq 4\,abcd.$$

This proves (ii); (iii) follows from (3), (4), and (5)

$$\begin{aligned}(a^2 + b^2)^2 + (c^2 + d^2)^2 &\geq (2ab)^2 + (2cd)^2 \\ &= 2(a^2b^2 + c^2d^2) + 2(a^2b^2 + c^2d^2) \\ &\geq 2(a^2b^2 + c^2d^2) + 4\,abcd = 2(ab+cd)^2.\end{aligned}$$

3. Suppose that m and n are positive integers with $m \geq n$. Show that the expression

$$\binom{m+n}{m} - \sum_{x=1}^{n} \binom{m}{x}\binom{n}{x}$$

is constant, and find its value, where $\binom{n}{r} = {}_nC_r$.

Solution 1. Consider the identity

$$(1 + x)^m (1 + x)^n = (1 + x)^{m+n}.$$

The coefficients of x^m in the binomial expansion of the RHS is $\binom{m+n}{m}$. Writing the LHS in the form

$$[1 + \binom{m}{1}x + \ldots + x^m][1 + \binom{n}{1}x + \ldots + x^n),$$

it is easily seen that the coefficients of x^m is

$$1 + \binom{m}{m-1}\binom{n}{1} + \ldots + \binom{m}{m-n}\binom{n}{n} = 1 + \sum_{i=1}^{n} \binom{m}{m-i}\binom{n}{i} = 1 + \sum_{i=1}^{n} \binom{m}{i}\binom{n}{i}.$$

It follows that $\binom{m+n}{n} = 1 + \sum_{i=1}^{n} \binom{m}{i}\binom{n}{i}$, or

$$\binom{m+n}{n} - \sum_{x=1}^{n} \binom{m}{x}\binom{n}{x} = 1.$$

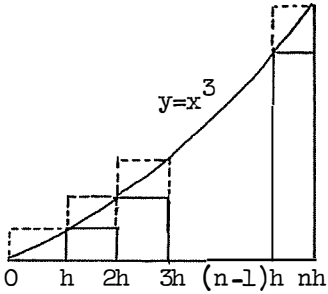
Solution 2: Take m boys and n girls. The number of m-person committees can be written down as $\binom{m+n}{m}$ or, using the "case method", as

$$\sum_{x=0}^{m} \binom{n}{x}\binom{m}{m-x} = \sum_{x=0}^{m} \binom{n}{x}\binom{m}{x} = 1 + \sum_{x=1}^{m} \binom{n}{x}\binom{m}{x}.$$

The result now follows.

4. Let A be the area of the region that lies under the graph of the curve $y = x^3$ and above the x-axis for x between zero and one. By dividing the interval $0 \le x \le 1$ of the x-axis into n equal parts and considering rectangles with one corner on the graph, show that for all positive integers

$$\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2} < A < \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$



Solution: Let $h = \frac{1}{n}$, and let A_i denote the area of the rectangle of width h and height $(ih)^3$. Referring to the diagram, it is immediate that

$$A_1 + A_2 + \dots + A_{n-1} < A < A_1 + A_2 + \dots + A_n.$$

Now

$$\begin{aligned} A_1 + A_2 + \dots + A_{n-1} &= h^4[1^3 + 2^3 + \dots + (n-1)^3] \\ &= \frac{1}{n^4} \cdot \frac{n^2(n-1)^2}{4} \\ &= \frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2} \end{aligned}$$

and

$$\begin{aligned} A_1 + \dots + A_{n-1} + A_n &= \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}\right) + h^4 n^3 \\ &= \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}. \end{aligned}$$

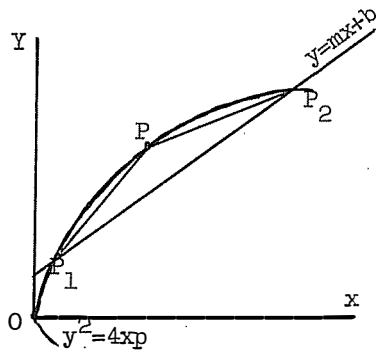
Thus

$$\frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2} < A < \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$

5. A chord $y = mx + b$ intersects a parabola $y^2 = 4px$ at P_1 and P_2. Prove that a third point P on the arc P_1P_2 of the parabola such that $\triangle PP_1P_2$ has maximum area is $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

Solution: Let the co-ordinates of P, P_1, P_2 (see diagram) be denoted by (x,y), (x_1,y_1), (x_2,y_2), respectively.
Then the area of the triangle P, P_1, P_2, is given by

$$A = \frac{1}{2} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$$



Using $x = y^2/4p$, et cetera, and expanding yields

$$A = y^2(y - y_2)/8p + y(x_2 - x_1)/2 + (x_1y_2 - x_2y_1)/2.$$

This is a quadratic of the form

$$A = -a y^2 + by + c, \quad a > 0, \; b > 0,$$

which has a maximum when $y = y_M = b/2a$. In this case $a = (y_2 - y_1)/8p$, $b = (x_2 - x_1)/2$, and

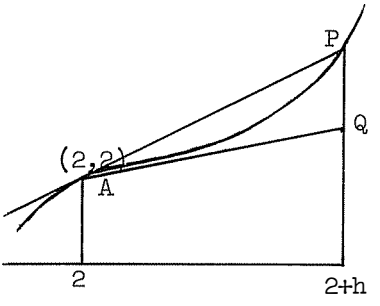
$$y_M = \frac{\frac{x_2 - x_1}{2}}{\frac{2(y_2 - y_1)}{8p}} = \frac{2p}{m},$$

since $m = (y_2 - y_1)/(x_2 - x_1)$. The point at which the area is maximum is thus $(p/m^2, 2p/m)$.

6. (a) Find the equation of the tangent to the curve $y = x^3 - 6x^2 + 9x$ at the point $(2,2)$.

(b) Show that the curve crosses the tangent at this point.

Solution: Let A denote the point $(2,2)$ and P the point on the curve with co-ordinates $(2+h, 2-3h+h^3)$. For h small, P is "close" to A on the curve. Since the slope of AP is $(-3h + h^3)/h = -3 + h^2$, the slope of the tangent at A is -3 (the limit of the slope of the secant AP as P approaches A). The equation of the tangent is thus



$$y - 2 = -3(x-2) \quad \text{or} \quad 3x + y = 8.$$

To prove (b), let Q denote the point on the tangent for which $x = 2 + h$. The co-ordinates of Q are $(2+h, 2-3h)$ so that $QP = (2-3h+h^3) - (2-3h) = h^3$. Thus, if $h > 0$, $QP > 0$ and P is above Q, whereas if $h < 0$, $QP < 0$ and P is below Q. It follows that the curve "crosses" the tangent at A.

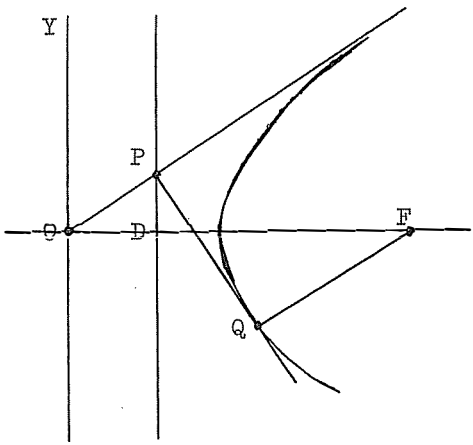
QUOTE OF THE MONTH

"There is no point in replacing old specialized topics by new ones, and the attempts to replace old outdated and musty subjects by new outdated and musty subjects (like boolean algebra) is not to be recommended."

Professor Serge Lang
Columbia University

7. The directrix of a hyperbola intersects an asymptote at P. From P a tangent meets the hyperbola at Q. If F is the focus corresponding to the directrix, prove that FQ is parallel to the asymptote.

Solution:



Let e denote the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The two directrices have equations $x = \pm \frac{a}{e}$, the corresponding foci have co-ordinates $(\pm ae, 0)$ and the two asymptotes have equations $\frac{x}{a} \pm \frac{y}{b} = 0$. Thus there are four cases to be considered, one of which is illustrated in the above diagram.

Rather than treat each case separately, let $\epsilon = \pm 1$, $\mu = \pm 1$, so that either directrix has equation $x = \epsilon a/e$, the corresponding focus F has co-ordinates $(\epsilon ae.0)$, and either asymptote has equation $y = \mu bx/a$. The point P then has co-ordinates $(\epsilon a/e, \epsilon\mu b/e)$; thus if Q has co-ordinates (x_1, y_1)

$$x_1(\epsilon a/e)/a^2 - y_1(\epsilon\mu b/e)/b^2 = 1.$$

Hence

$$(1) \qquad a\, y_1 = \mu b\, x_1 + \mu\, \epsilon abe.$$

Now the slope of FQ is $\frac{y_1}{x_1 - \epsilon ae}$, the slope of the asymptote is $\mu\, b/a$, and

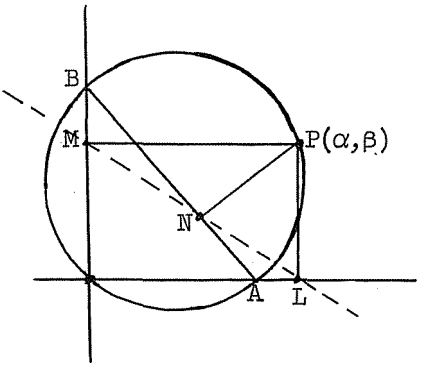
$$\frac{y_1}{x_1 - \epsilon ae} - \mu\frac{b}{a} = \frac{ay_1 - \mu b\, x_1 - \mu\, \epsilon abe}{a(x_1 - \epsilon ae)} = 0$$

by (1); hence FQ is parallel to the asymptote.

8. The circle $x^2 + y^2 - ax - ay = 0$ passes through the origin and also intersects the x and y axes at A and B, respectively. From any point P on the circle perpendiculars are drawn

to meet the x axis at L, the y axis at M, and AB at N. Prove that L, M, N, are collinear.

Solution:



The co-ordinates of A and B are (a,0) and (0,a) respectively so that the equation of AB is $x + y = a$. Let P have co-ordinates (α,β) so that

(1) $\alpha^2 + \beta^2 - a\alpha - a\beta = 0$,

and the co-ordinates of L and M are $(\alpha,0)$ and $(0,\beta)$ respectively. The point N must satisfy the equation of LM, that is, $\frac{x}{\alpha} + \frac{y}{\beta} = 1$. Since NP is perpendicular to AB, the co-ordinates of N have the form $(\alpha + k, \beta + k)$ for some k. Since N lies on AB, $(\alpha + k) + (\beta + k) = a$ or $k = (a - \alpha - \beta)/\alpha$, and N has co-ordinates $[\frac{1}{2}(a + \alpha - \beta), \frac{1}{2}(a - \alpha + \beta)]$.

Furthermore, N lies on LM since

$$\frac{\frac{1}{2}(a + \alpha - \beta)}{\alpha} + \frac{\frac{1}{2}(a - \alpha + \beta)}{\beta} - 1$$

$$= \frac{a\alpha + a\beta - \alpha^2 - \beta^2}{2\alpha\beta} = 0$$

by (1) (provided $\alpha\beta \neq 0$; the points are obviously collinear in this special case).

9. (a) Prove that the distance d between the circumcentre and and the incentre of a triangle ABC is given by $d^2 = R^2 - 2Rr$, where R and r are the circumradius and the inradius of the triangle, respectively.

(b) If the circumcentre is on the inscribed circle, prove that

$$\cos A + \cos B + \cos C = \sqrt{2}.$$

Solution: Part (a) is taken directly from Hall and Knight's "Elementary Trigonometry", p.223.

To prove part (b), two other deductions may be used; on page 128 of Hall and Knight we find that, for any triangle (use conversion of sums to products)

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

On page 223,

$$\frac{r}{R} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

It follows that

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

The required identity is now easy to obtain. Since the circumcentre is on the inscribed circle $d^2 = r^2$, so that $r^2 = R^2 - 2Rr$, and if we complete the square $(r - R)^2 = 2R^2$ or $\frac{r - R}{R} = \sqrt{2}$. Further, $R^2 - r^2 = (R - r)(R + r) = 2Rr$, implies

$$\frac{r - R}{R} = \frac{2r}{R + r} = \sqrt{2} \quad \text{or}$$

$$\frac{R + r}{r} = 1 + \frac{R}{r} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

This proves the formula.

It is the opinion of the Problems Editor that 9(a) would have been sufficient for a problem.

10. If a, b, and h are constants, prove that the maximum and minimum values of

$$F(\theta) = a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta$$

are the roots of the equation

$$(x - a)(x - b) = h^2.$$

Solution:

Set $F(\theta) = x$, $\tan\theta = t$, so that

$$\cos^2\theta = \frac{1}{1 + t^2}, \quad \sin\theta \cos\theta = \frac{t}{1 + t^2}, \quad \sin^2\theta = \frac{t^2}{1 + t^2}.$$

Substituting in the given expression, we get

$$(a - x) + 2ht + (b - x)t^2 = 0.$$

Since all real values of t are possible, it follows that the discriminant

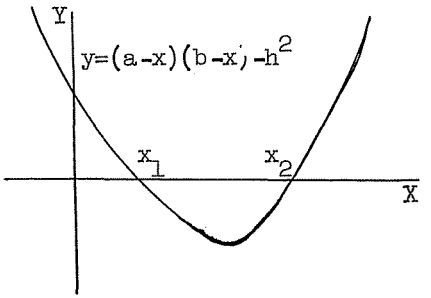
$$(2h)^2 - 4(a - x)(b - x)$$

must be positive, or equivalently

$$(a - x)(b - x) - h^2 < 0.$$

The graph of

$$y = (a - x)(b - x) - h^2$$



is a parabola opening upwards so that $y < 0$ only between the roots x_1, x_2, as shown in the diagram. It follows that the maximum and minimum values of x are x_2 and x_1 respectively.

11. The lengths of the sides of a quadrilateral are a, b, c, and d; the sum of a pair of opposite angles is ω. If the sides are tangents to a circle, prove that the area of the quadrilateral is given by

$$\Delta = \sqrt{abcd}\,\sin\frac{\omega}{2}\,.$$

Solution:

Since a circle can be inscribed in the quadrilateral, the sum of both pairs of opposite sides is the same, that is, $a + c = b + d$. Setting $\sigma = \frac{1}{2}(a + b + c + d)$, it follows that $\sigma = a + c = b + d$, or

(1) $\sigma - a = c$, $\sigma - b = d$, $\sigma - c = a$, $\sigma - d = b$.

Now the area Δ of any quadrilateral is given by

(2) $\Delta^2 = (\sigma - a)(\sigma - b)(\sigma - c)(\sigma - d) - abcd\cos^2\frac{\omega}{2}$

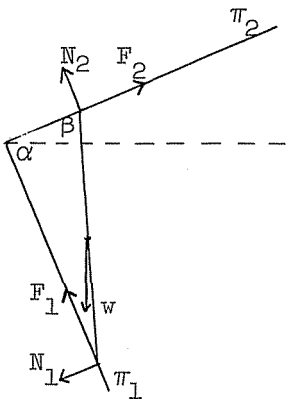
[Reference, Hall and Knight, p.229]; hence, using (1),

$$\Delta^2 = abcd - abcd\cos^2\frac{\omega}{2} = abcd\sin^2\frac{\omega}{2}\,;$$

thus, $\Delta = \sqrt{abcd}\,\sin(\omega/2)$.

12. Let π_1 and π_2 represent two mutually perpendicular planes whose line of intersection is horizontal. Plane π_1 is inclined at an angle α to the horizontal. A uniform rod is placed with its ends in contact with the planes, the rod being perpendicular to their line of intersection. The coefficient of friction between the ends of the rod and the planes is μ. In an equilibrium position, prove that the rod makes an angle $\alpha + 2\theta$ with the plane π_2, where $\theta = \tan^{-1}\mu$.

Solution: Let T_1, T_2, denote the tangential forces at the points of contact of the rod and the planes π_1, π_2; let N_1, N_2, denote the normal reactions at the points of contact as shown in the diagram. Let β denote the angle the rod makes with π_2,



let w denote the weight of the rod, and let 2d denote the length of the rod. Since the rod is uniform, the weight acts downward through the centre of the rod as shown.

Equating the components of the forces in the directions of the two planes, yields

(1) $N_1 - F_2 + w \cos \alpha = 0,$

(2) $N_2 + F_1 - w \sin \alpha = 0.$

Taking moments about the centre of the rod yields

(3) $dN_2 \cos \beta - dF_2 \sin \beta - dN_1 \sin \beta - dF_1 \cos \beta = 0.$

Finally, the definition of μ implies

(4) $N_1 = F_1\mu, \quad N_2 = F_2\mu.$

The five equations (1), (2), (3), (4), involve the five unknowns F_1, N_1, F_2, N_2, and w. Notice that d can be cancelled in (3) homogeneously and thus may be eliminated to obtain the required result. One way of doing this is the following. Eliminate w from equations (1) and (2) by subtraction, and substitute (4) in the resulting expression, and in equation (3). This yields

(5) $F_1 (\cos \alpha - \mu \sin \alpha) + F_2(\mu \cos \alpha + \sin \alpha) = 0$

and

(6) $F_1 (\cos \beta + \mu \sin \beta) + F_2(- \mu \cos \beta + \sin \beta) = 0.$

Finally, eliminating F_1, F_2 from (5) and (6) (equate expressions for F_1/F_2 or use determinants) to obtain

$$\mu^2 (\cos \alpha \sin \beta - \cos \beta \sin \alpha) + 2\mu(\cos \alpha \cos \beta + \sin \alpha \sin \beta) + \sin \alpha \cos \beta - \sin \beta \cos \alpha = 0$$

or

(7) $(\mu^2 - 1) \sin (\beta - \alpha) + 2\mu \cos (\beta - \alpha) = 0.$

Finally, from (7)

$$\tan (\beta - \alpha) = \frac{2\mu}{1 - \mu^2} = \frac{2 \tan \theta}{1 - \tan^2\theta} = \tan 2\theta$$

which implies $\beta - \alpha = 2\theta$ (since angles are acute) or $\beta = \alpha + 2\theta$ as required.

Further information on District 10 Mathematics Contest

Ron Dunkley, St. Mary's

Since the October issue of the Gazette, much progress has been made in preparation for this year's contest. The Committee is happy to be able to report that chairmen have been appointed for all except one district, We are optimistic that this one district will have a chairman by the time this article is read. Indications at the present time are that there will be some 400 schools participating, with approximately 5,000 students competing.

We are also happy to report that representatives have been added to the original committee, from the Toronto area, Hamilton, and London, in an attempt to ensure representation to other districts than District 10, in order to provide the very best test paper possible.

By the time this article reaches you, there should be an entry from your school. If you have not received one and are interested in entering the Contest, please contact Ron Dunkley, at St. Mary's District Collegiate Institute, St. Marys. We extend a special invitation to any private schools or separate schools not affiliated with the O.S.S.T.F. We will be happy to receive your entry, and will assign you to an O.S.S.T.F. district for contest purposes.

In the October issue of the Gazette, a list of district chairmen was published. Below are the names of those who have accepted positions since that time. We of the Committee wish to extend our thanks to these chairmen who have willingly accepted these posts.

District 3: W. A. Strang, Sarnia Central C.I.

District 5: John Beamer, Port Dover Composite School

District 15: C. Walton, Leaside High School

District 16: D. Clarke, Hill Park S.S.

District 18: G. Kaye, Downsview S.S.

District 20: I. Parkinson, Port Perry H.S.

District 21: Jack Dahmer, Park Street C.I., Orillia

District 22: J. Staples, Lindsay D.H.S.

District 23: L. Chouinard, Napanee D.S.S.

District 28: F. Donnelly, Sault Ste. Marie C.I.

District 29: Beryl Tindall, Kirkland Lake C. and V.I.

BOOK REVIEW

Mathematics-A New Approach, by D.R.Mansfield and D.Thompson:Book I
Clarke,Irwin and Co.Ltd., Toronto, 1962

Pupil's Book 142 pp, clothbound - $1.45
Teacher's Book 109 pp, " - $1.80

Pupil's Book:

This is the first of a series of three texts based on teaching material used in Holloway School in Britain.

The authors believe at the 11-plus stage of a child's development his change in school should be accompanied by a change in curriculum. This text has been written in an attempt to satisfy this end in mathematics. In this approach less material is covered than in the traditional course used for the regular Grammar School standard. Topics are introduced in a rather novel way and material usually studied at a much higher academic level is introduced in a simple way.

The material of this book has been used with good results (according to the preface) with students ranging from backward to highly intelligent.

Discussion of the text material,chapter by chapter,follows.

The natural number system, with a review of the four basic arithmetical operations, is covered in chapter one but is introduced through the binary number system with reference to electronic computer applications. This change in approach during the revision should help to overcome the disappointment and depression at the further review.

Chapter two deals with the common properties of points, lines, surfaces, and angles; these Euclidean ideas are discussed in comparison with non-Euclidean geometries and through the use of Rectangular Cartesian co-ordinates in graphing of diagrams. Some exercise material introduces three-dimensional figures and intersection of planes in simple cases.

Fractions, mixed numbers, and percentages are the topics of chapter three. These are introduced through the historical necessi to extend the number system to obtain solutions to such problems as:

> "What is the length of each side of a square building if the distance right round it is forty-nine paces?" or
> "What number is such that if we multiply by three we obtain eight?"

Some importance is attached to the idea of listing fractions in ascending order up to some given denomination through the use of point lattices thus leading to Farey Series. The study of fractions is extended through complex fractions to continued fractions.

The following chapter applies fractions and percentages to work with statistics. Statistical methods of collection, compilation, and graphical presentation in various forms are practised.

Prime numbers, factoring, and index notation are treated in another chapter. Some emphasis is given to the listing of all primes up to a given number using the Sieve of Eratosthenes. Students in the exercises are encouraged to practice with Fermat's and Mersenne's attempts at expressing large prime numbers.

Measurement, tables, and conversions in both the British and Metric systems are dealt with briefly.

A basic approach is made to algebra with considerable discussion and practice with commutative versus non-commutative algebras (using rotations and translations of geometric figures). Simplification of algebraic expressions, manipulation of formulae, and solution of equations are dealt with.

The final chapter includes a study of geometry: rectilinear figures and their classification, triangles classified according to sides and angles, calculation of areas and volumes, discussion of symmetry and non-symmetry, sections and tesselations.

Each chapter includes a great deal of practice material; in fact approximately fifty percent of the text is exercise material. Exercises are well graded with only a few rather difficult examples included in each. The explanatory material is well illustrated (both by word and picture) in a meaningful way using simple examples. Concrete examples referring to specific applications of the material tend to make the exercises more meaningful.

The text does not include answers to the exercises, nor does it include an index.

Teacher's Book 1:

The teacher's book contains some general notes dealing with the purpose of the material in that chapter, reasons for its inclusion in the courses, and possible pitfalls for the student.

Suggestions are included in each chapter for extra reading both for students and teacher.

Special emphasis is given to certain sections of the chapter or to certain special or important problems in the exercise material.

Answers for each problem of the exercise material are included as well as numerous suggestions for further study.

J.F.Chambers, Mathematics Department,
Jarvis Collegiate Institute, Toronto.

Dr. M. F. Rosskopf

Main Speaker at 1964 Easter Convention of the O.A.T.M. and P.

Garth Kaye, Downsview

In keeping with the policy of having a leading mathematics educator as the main speaker at the Easter Convention of the Ontario Association of Teachers of Mathematics and Physics, the executive has arranged to have Dr. Myron F. Rosskopf, Professor of Mathematics, Teachers College, Columbia University, New York, give two lectures during this year's meeting.

Dr. Rosskopf obtained the degree of Ph.D. from Brown University in 1934. From 1934 until 1952, except for three years in the U.S. Air Force, he was Head of the Department of Mathematics at John Burroughs School in Clayton, Mo. In 1952 he was appointed to the staff of Columbia Teachers College; since 1956, he has held the position of Professor of Mathematics there.

Many articles written by Dr. Rosskopf have been published, over the years, in "The Mathematics Teacher" and "School Science and Mathematics". In addition to these articles, Dr. Rosskopf is co-author of ten text books including "Logic in Elementary Mathematics" and "Modern Arithmetic Through Discovery, Grades 3-6". As Educational Consultant for the McGraw-Hill Book Company, he also assisted in the preparation of twelve filmstrips dealing with the teaching of mathematics.

In addition to his published papers and books, Dr. Rosskopf has spoken to many groups of mathematics teachers in several states. He has acted as a consultant to the Eight Year Study of the Progressive Education Association, New York State Department of Education, and to individual schools. He was a member of the Mathematics Regents Examination Committee for a period of two years. Dr. Rosskopf was active in the organization of the Association of Mathematics Teachers of New York State and served as president during the 1953-54 term. He was editor of the New York State Mathematics Teachers' Journal, 1954-56; Associate Editor of the Mathematics Teacher, 1953-59; and editor of the Mathematics Student Journal, 1961-64.

At the suggestion of a number of members of the O.A.T.M. and P., Dr. Rosskopf has been asked to give a talk entitled "Where Must Euclid Go?, which he delivered at the N.C.T.M. Convention in Pittsburgh last spring. His second address will deal with Mathematical Structures. In view of the changes taking place at present in mathematics curricula in Ontario, both of these addresses will be particularly timely for Ontario Mathematics Teachers.

Why should I belong to the Ontario Association of Teachers of Mathematics and Physics?

Garth Kaye, Downsview

This question has been asked many times by Ontario Mathematics teachers. For the teacher who is concerned only with direct personal benefits from membership, any answer that can be given will be unsatisfactory; but for the teacher who is seriously concerned about the state of mathematics education in the province, there are numerous reasons why the Association deserves his support.

As a section of the O.E.A., the O.A.T.M. and P. has representation on the Board of Directors. This representative is expected to speak on behalf of the organization and mathematics teachers in general. Only if the views of mathematics teachers are made known to the Executive and Council of the O.A.T.M. and P. can this representation be effective.

Of great significance is the appointment by the O.A.T.M. and P. of three members to the Ontario Mathematics Commission. In view of the significant contributions the Ontario Mathematics Commission has made and is making to mathematics curriculum development, and in view of the possible association of the O.M.C. with the newly formed Curriculum Institute, the representation of the O.A.T.M. and P. on the Commission is of prime importance to the teachers of Mathematics in Ontario.

The two factors which place the heaviest drain on the limited finances of the O.A.T.M. and P. are the Ontario Mathematics Gazette and the annual convention of the Association at Easter each year.

The Gazette is rapidly becoming recognized as a useful, informative magazine for mathematics teachers. Members of the O.A.T.M. and P. can well be proud that their organization is associated with this venture.

Each year a great amount of time and energy is necessary for the planning of the annual convention and each year the number of members attending increases, producing more problems of accommodation and planning. The Executive and Council of the O.A.T.M. and P. attempts to provide a programme that will be of interest to all teachers of Mathematics in the province.

The Association is anxious to help, wherever it can, in the formation of Seminar groups, similar to those operating in a few parts of the province at the present time. As an affiliate of the National Council of Teachers of Mathematics, the O.A.T.M. and P. is able to make use of the extensive services of that organization to this end.

One of the biggest problems facing the Association is the problem of discovering the most effective ways to be of assistance to the membership. A committee is at work, this winter, drafting constitutional changes which, it is hoped, will help in this

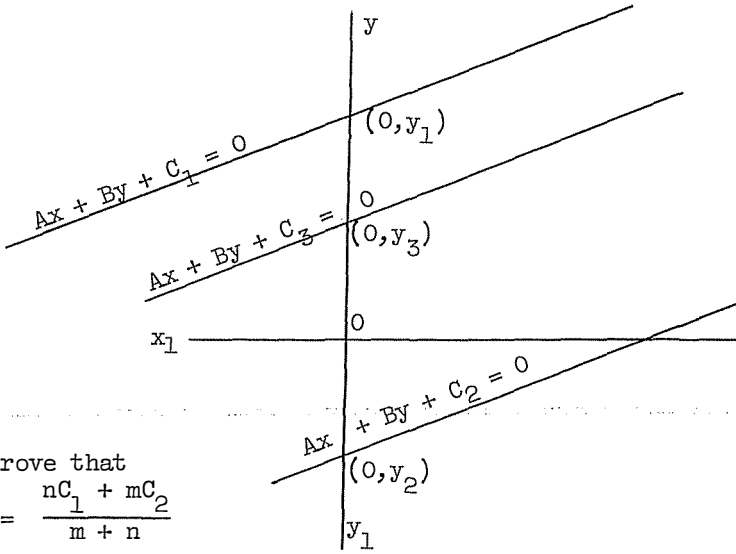
regard. Constitutional changes can only provide a better structure in which to work. It is the members who can help the Association to be more effective by making known to the Executive and Council their suggestions and criticisms.

The O.A.T.M. and P. needs your support and membership if it is to be more effective in its rôle as the only province-wide association of teachers of mathematics.

A PROBLEM ON PARALLEL LINES

Dawn Crawford 13A,
Napanee District Secondary School

Given: Three parallel lines with equations $Ax + By + C_1 = 0$, $Ax + By + C_2 = 0$, and $Ax + By + C_3 = 0$ such that the line with equation $Ax + By + C_3 = 0$ is between the other two lines and divides the distance between them in the ratio m to n



Required: to prove that

$$C_3 = \frac{nC_1 + mC_2}{m + n}$$

Proof: Let the points where the three lines meet the y-axis be $P_1(0,y_1)$, $P_2(0,y_2)$, and $P_3(0,y_3)$.

These points cut segments from the y-axis in the ratio m:n (similar triangles). P_1 is on the line with equation $Ax + By + C_1 = 0$; hence

(1) $\quad 0 + By_1 + C_1 = 0$

(2) $\quad 0 + By_2 + C_2 = 0$

(3) $\quad 0 + By_3 + C_3 = 0$

By a previous theorem, $\dfrac{my_1 + ny_2}{m + n} = y_3$

Multiply (1) by n : $Bny_1 + nC_1 = 0 \qquad (5)$

Multiply (2) by m : $Bmy_2 + mC_2 = 0 \qquad (6)$

Add (5) and (6): $B(ny_1 + my_2) + nC_1 + mC_2 = 0$

Divide by $m + n$ (7) $\dfrac{B(ny_1 + my_2)}{m + n} + \dfrac{nC_1 + mC_2}{m + n} = 0$

$$By_3 + \frac{nC_1 + mC_2}{m + n} = 0$$

But $By_3 + C_3 = 0$

Therefore, $C_3 = \dfrac{nC_1 + mC_2}{m + n}$

Problem 7. Given $\triangle ABC$ and median AD. P is <u>any</u> point on AD. BP is produced to meet AC at F and CP is produced to meet AB at E. Prove EF || BC.

Solution by D.E.Scovell, David and Mary Thomson C.I., Scarborough.

Since AD, BF, and CE are concurrent, by Ceva's theorem *

$$BD \cdot CF \cdot AE = DC \cdot FA \cdot EB$$

But AD is a median; hence BD = DC

Thus $CF \cdot AE = FA \cdot EB$

that is, $\dfrac{AE}{EB} = \dfrac{FA}{CF}$

Hence EF || BC (IV-3)

* See "A Modern Geometry for High Schools", Lougheed, Workman, Prop V-13, or "Deductive Geometry", P.A.Petrie, Question No. 18, Rev. Ex. 87.

THE GRADE 13 ALGEBRA PAPER

by the Examiner in Algebra

[See also the committee comments distributed to all teachers]

1. (a) Standard theory; well done. Students tended to omit + signs and dots in

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}.$$

These are essential for proper form. The formula must be stated as "$S_n = (a-ar^n)/(1-r)$, PROVIDED $r \neq 1$".

1. (b) Very badly answered, especially the theory in 1 b(i). Among the many incorrect statements occurring were ones such as: (1) when $|r| > 1$, then r^n increases as n increases, et cetera. One possible answer follows.

(i) If $|r| < 1$, then, as n becomes arbitrarily large, r^n approaches zero. Thus, $1-r^n$ approaches 1, and

$$S_\infty = \lim_{n\to\infty} S_n = a/(1-r).$$

(ii) If $|r| > 1$, then, as n becomes arbitrarily large, $|r^n|$ also becomes arbitrarily large. Then $|1-r^n|$ becomes arbitrarily large, and so the quantity $S_n = a(1-r^n)/(1-r)$ becomes infinite. Thus S_∞ does not exist.

(iii) If $|r| = 1$, then $r = 1$ or $r = -1$. If $r = 1$, $S_n = na$; $n \to \infty$, $|S_n|$ becomes arbitrarily large and S_∞ does not exist. [Note: The formula $S_n = a(1 - r^n)/(1 - r)$ can NOT be used here, since it was developed on the assumption $r \neq 1$.]

If $r = -1$, $S_n = 0$ or a according as n is even or odd; thus, as $n \to \infty$, S_n keeps oscillating between 0 and a, and does not approach a limit. Hence S_∞ does not exist.

Too many students prattled along lines such as: "In the formula $S_\infty = a/(1-r)$, the denominator is zero if $|r| = 1$ and is negative if $|r| > 1$; hence S_∞ does not exist". This statement is incorrect per se, but it is especially blameworthy to try to use the formula $S_\infty = a/(1 - r)$, which was developed on the assumption that $|r| < 1$, in those cases where it does not apply.

Too many students claimed, in 1(b)i, that the series possessed a sum to infinity because successive terms in the series were steadily approaching zero in magnitude. This fallacious argument would also "prove" that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$$

possessed a sum to infinity!

Some regard must be paid to grammar and sense. We could not award marks for the "sentence": "As $1-r^n \longrightarrow 0$ as $n \longrightarrow \infty$ as

$r^n \longrightarrow 0$" because we could attach no meaning to it.

The words "negligible" and "discard" are not mathematical terms. It is wrong to say: "In $S_n = a(1-r^n)/(1-r)$, for large values of n, the value of r^n is so small that it is negligible and can be discarded".

3. (b) A fundamental weakness showed up here. Square completion at once gives $p = 2$, $q = 3/4$, $r = 31/8$. Too many students then spent a page in finding the minimum by using the "formula" or the "discriminant" rather than note that "$2(x + 3/4)^2 \geq 0$; hence the minimum value is $31/8$, and is taken when $x = -3/4$".

The "formula method" led to so much refusal to think that I feel it would be better to refrain from teaching it at all; the same remark probably applies to the "discriminant method".

Too many students stated incorrectly that "$(x + 3/4)^2$ is always positive for x real". Those who obtained p, q, and r by comparing coefficients sometimes neglected to point out that, to have an <u>identity</u>, the coefficients of like powers of x on the left and right hand sides must be equal.

Square completion is a technique of enormous importance, but it must be <u>understood</u>.

6. (a) A series is not equal to its first 3 terms.

$$(4 + x)^{\frac{1}{2}} = 2(1 + x/4)^{\frac{1}{2}} = 2 + x/4 - x^2/64 + \ldots$$

$$(1 + 3x)^{-1/3} = 1 - x + 2x^2 + \ldots$$

Do not omit dots.

(b) Solution: The expansion of $(1 + x/4)^{\frac{1}{2}}$ is valid for $|x| < 4$. The expansion of $(1 + 3x)^{-1/3}$ is valid for $|x| < 1/3$. Hence, the final expansion is valid for $|x| < 1/3$. One common error was to fail to identify the series mentioned; statements occurred such as "it is valid for $|x| < 4$, and it is valid for $|x| < 1/3$; so it is valid for $|x| < 1/3$".

Another error: "$\sqrt{4 + x}$ is valid for $|x| < 4$". Note that $x = 21$ may be substituted in $\sqrt{4 + x}$; it is the <u>series expansion</u> that imposes the restriction $|x| < 4$.

A few students used meaningless symbolism such as $-\frac{1}{3} \leq x \geq \frac{1}{3}$.

7. (a)

$$f(x) = \frac{1}{2 + (x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2} = \frac{1}{x + x^{-1}} = \frac{1}{x + 1/x}.$$

Note that $f(0)$ is not defined, since division by zero is inadmissible. Hence

$$f(x) = \frac{x}{x^2 + 1} \quad (x \neq 0), \qquad g(x) = \frac{x}{x^2 + 1}$$

Thus $f(x) = g(x)$ for all real x <u>except</u> $x = 0$. The exception was usually missed. We were prepared to allow the answer "$f(x) = g(x)$ for all real $x > 0$" in case any students objected to working with complex quantities in intermediate steps

[as, for instance, if one tried to compute $f(-2)$].

However, this solution did not appear.

One comment on 7 (a) must be made. Many students insisted on setting $y = x/(x^2 + 1)$ and then applying the "discriminant method" blindly. This is another reason for omitting this "method". It is not a good way of finding maxima and minima of rational fractional functions; and students seem to want to "plug it in" at every turn.

9. (a) Answer: ${}_{12}C_5 = 792$. [One solution stated "The 5 kneaded people can be chosen in 792 ways".]

A common clumsy solution was by "cases" to give:

$$ {}_8C_5 + {}_8C_{4\cdot 4}\,{}_1C_1 + {}_8C_{3\cdot 4}\,{}_2C_2 + {}_8C_{2\cdot 4}\,{}_3C_3 + {}_8C_{1\cdot 4}\,{}_4C_4 = 792. $$

Students should pay attention to the fundamental hypergeometric identity (easily proved by taking committees of n from a men and b women):

$$ {}_{a+b}C_n = {}_aC_n \cdot {}_bC_0 + {}_aC_{n-1} \cdot {}_bC_1 + {}_aC_{n-2} \cdot {}_bC_2 + \cdots $$

$$ + {}_aC_2 \cdot {}_bC_{n-2} + {}_aC_1 \cdot {}_bC_{n-1} + {}_aC_0 \cdot {}_bC_n. $$

It is to be hoped that we soon can eliminate obsolete notations in Ontario and always write $n!$ instead of $\lfloor n$, $\binom{n}{r}$ instead of ${}_nC_r$.

Final Notes: (1) Students who insist on showing off their "extra knowledge" should make sure they know what they are talking about. Otherwise they may perpetrate absurdities such as the following remark in 7(a). The student explained that he used R to represent the set of real numbers, C to represent the set of complex numbers, and then wrote the wondrous explanation "$x^2 + 1 = 0$ when $x \in C$ $\therefore$ $C \notin R$".

Another "statement", appearing in 4(b), was: "As x gets bigger, this curve get closer and closer to the null set Φ, but never reaches it".

(2) Many students lose marks by fishing up a "corresponding equation" every time they see a function. This occurred frequently in questions 3, 4, 7. It is unfortunate that this needless and misleading term was ever introduced.

(3) Students should give complete solutions to problems. Marks can not be given for correct steps not shown if all the student does is record an incorrect result.

[Modern non-Mathematics - continued from page 37]

There is very little that a single short article can do in such a cause, but the writer does hope that this little note might at leas stimulate others to assist in putting non-math where it rightly belon Appearances to the contrary, it is not yet universally admired by mat maticians, and more extensive discussion might remove the many under-standings which apparently exist.

WHAT IS A FUNCTION?

1. Introduction.

Recently there has been considerable interest in defining a function. An initial over-emphasis on the "ordered pairs" approach has led some educators to the point of view put forth by C. B. Allendoerfer [American Mathematical Monthly, June-July, 1962]: "Have we really improved the clarity of the idea of a function when we define it as a 'set of ordered pairs, etc.' when we might with equal rigor define it in terms of its domain, range, and rule?" In this article, we shall sketch some aspects of the development of the functional concept, and will indicate our own view that the "ordered pair" approach is philosophically unsound and will lead the student to difficulty later on; these difficulties need not be introduced if one thinks of a function as a mapping or correspondence.

If a symbol x may assume arbitrary values chosen from a set R of numbers, then we refer to x as an independent variable; in elementary work, x may usually assume only real values and, unless contrary information is given, we shall always consider x as a real independent variable. If x is restricted in any way, the information may be supplied; for example, $x > 3$. Naturally, the letter x is not necessarily the symbol used to denote an independent variable [for further discussion of functions and variables, see Topics in Modern Mathematics edited by R.G.Stanton and K.D.Fryer, Chapter 7; Prentice-Hall, 1964].

"If to any real number x there corresponds a certain number $y = f(x)$, then we say that a function f is defined over the set of real numbers."

We have quoted this definition from K. Kuratowski, Introduction to Calculus, Addison-Wesley Publishing Company, 1962. Let us pursue this definition a little further and generalize it. First, we need a set of numbers from which the values of x may be selected; this set is the domain of the variable x. Usually in high school, the domain of the variable x is the set of real numbers. Then we need a method of assigning to certain values of x, in the domain of the variable, corresponding values of y; this method is usually given by an algebraic rule, for example, $y = 3/x$. In this example, we notice that if $x = 4$, then $y = .75$. The rule enables us to assign one value y to each real number x, with the exception of $x = 0$. So we say that the rule defines a function f; the domain of the function f consists of all real numbers $x \neq 0$. Notice the distinction between the domain of the variable x and the domain of the function f; we can try any real value of x, but the domain of f consists only of those values of x which produce a value for y.

Formalizing the remarks of the preceding paragraph, we might state: If a second quantity y is such that, given a subset S of values of x, where S is contained in R, there is determined exactly one value for y, then we say that y is a dependent variable. We further state that y is a function of x. The values of y will range over a set of numbers; in elementary

work, we frequently restrict these values to the set of real numbers. Then we can say that y is a real function of the real variable x. The set S is the domain of the function f; the set of values assumed by y is called the range of the function f.

Example: y is defined as a real function of the real variable x by the rule

$$y = \sqrt{9 - x}\ .$$

We have thus defined a function f which makes one y-value correspond to certain x-values. The domain of f is the set of all real numbers less than or qual to 9; the range of f is the set of all real numbers greater than or equal to 0. The function f sets up a correspondence in the following way:

$$-7 \xrightarrow{\ f\ } 4$$

Read: The function f maps the real number -7 on the real number 4 (sometimes it is handy to think of the function f as a distorting mirror which takes the "object" -7 and produces the "image" 4).

In concluding this section, we should make two remarks. First, we have not yet really defined a function; we have defined what is meant by the concept of "being a function of", and this is the really important concept. We know that one quantity y "is a function of" a quantity x if, given an x-value in the domain of the function f, we can obtain a corresponding y-value. Secondly, we have restricted ourselves to speak only of single-valued functions; it is possible to define "multiple-valued functions" where a whole set of y-values corresponds to a given x-value. However, such a definition introduces ambiguities and complications, and will not be mentioned further in this article.

2. Definition of a Function by Prescribing its Domain, Range, and Rule.

Let us be given the rule $y = x^2 - 2x + 7$.

Since no further information is given, we take x as an arbitrary real number [recall that we agreed that x should be real unless otherwise stated]. Since any real x produces a value for y, the rule defines a function f whose domain is the entire set of real numbers. On the other hand, since we may write

$$y = (x - 1)^2 + 6,$$

we see that the range of f is the set of all real numbers greater than or equal to 6.

Now just what is the function f in the above example? We have seen that the function f is determined by the rule

$$y = x^2 - 2x + 7.$$

Indeed, some textbooks state that the function f is the given rule [assuming that we know that the domain and range of f are real]. This method of defining a function is not highly sophisticated, but is not likely to lead to any difficulty. Perhaps it is

even the best method for many classes. This method is employed in A First Course in Calculus by Serge Lang [Addison Wesley, 1964], where we find the perfectly workable definition: A function is a rule which to any given number associates another number.

However, there are more abstruse ways of looking at the functional concept. We can agree that the rule

$$y = x^2 - 2x + 7$$

defines a function f; now we shall consider other ways of looking at the functional concept, and will look at the function f as the procedure whereby given $x = 5$, we make correspond the y-value 22; given $x = 2$, we make correspond the number 7, et cetera.

3. Definition of a Function as a Set of Ordered Pairs.

Let us pursue a little further the example of the last section. We saw that the rule

$$y = x^2 - 2x + 7$$

allowed us to compute pairs of corresponding values of x and y. Thus for $x = 1$, we find that $y = 6$; for $x = 3$, we find that $y = 10$; etc. If we agree to write x first and y second, the given rule allows us to construct an arbitrary number of ordered pairs, namely,

$$(1,6), \quad (2,7), \quad (3,10), \quad (.5, 6.25), \quad \text{etc.}$$

It has become quite fashionable in some quarters to define the function f as the infinite set of all such ordered pairs.

This definition is not, to my way of thinking, philosophically sound. In order to build up the ordered pairs, we must normally be given the rule of procedure; then we might as well use the definition given in Section 2 and define f as the "rule". But there are graver objections than the fact that the "ordered pair" definition is rather derivative or secondary in its nature. We might complain that it is impossible to write down all the ordered pairs; however, we could avoid this difficulty by writing down a "typical ordered pair". Such a typical ordered pair would be $(x, x^2 - 2x + 7)$; but then we are right back at the "rule" definition of a function, since we have given a rule for finding the y corresponding to any x!

It thus appears that the "ordered pair definition" is just a disguised version of the "rule" definition, but that it suffers from being one step further removed from being basic.

[An interesting question which might be put to proponents of the "ordered pair" approach is this: How does one define an "ordered pair" without prior introduction of a function f whose domain is the integers 1 and 2?.]

4. A Function is not a Set of Ordered Pairs

Let us take three "homely" situations which may be of use to students puzzling over the concept of "function".

1. You go to a country fair where there is a rather strange distorting mirror. Whenever you hold up a card on which a real number is written, the mirror makes you see the square of the number. Thus, if the "object" is 4, the "image" is 16; if the object is -2.2, the image is 4.84.

The object number is then the independent variable x; it is the number which you arbitrarily assign. The image number is the dependent variable y; it is the number which you see in the mirror.

The function f is the correspondence between object and image; in a way, one could say that f is the mirror which takes the number x and turns it into the number x^2.

We can write

$$x \xrightarrow{\quad f \quad} x^2,$$

and say: f is the function whose value at x is x^2. We also write $f(x) = x^2$, and say: the function f operates on x to give x^2.

2. We may think of the last situation, but think of our function as a meat-grinder. Whenever we throw a number into the grinder, another number comes out. If we throw in the number -2, the number 4 comes out; if we throw in the number x, the number x^2 comes out. In this case, we can think of the function as being the meat-grinder; f operates on the number x to produce x^2.

3. Finally, a more instructive example. Suppose that we consider the set of all married couples in the city of Toronto [we assume that all these people live in Toronto and that there is no bigamy]. Then we can think of a function f which makes correspond to any person in this set another person (his spouse); we shall call this function f <u>marriage.</u>

Now f allows us to know $f(x)$, given x; for example,

f(Mr. John Smith) = Mrs. Elaine Smith,
f(Mrs.Beatrice Jones) = Mr. James Jones,
etc.

Now this function f is the marriage ceremony; it has set up a correspondence between pairs of people and it has produced a "set of ordered pairs". But the set of ordered pairs, in this case the set of married couples, is the result of applying f. It is not f.

No one would claim that the marriage ceremony was identical with the set of married couples resulting from it. Similarly, we should not confuse the function f with the set of ordered pairs resulting from the use of the functional operator f.

5. Definition of a Function as a Mapping or Correspondence.

Suppose that a function f is defined by the rule:

$$f(x) = x^3 + 5.$$

We should then think of f as a correspondence which sets up an association in the following way:

$$2 \xrightarrow{f} 13,$$

$$-3 \xrightarrow{f} -22,$$

$$0 \xrightarrow{f} 5,$$

and, in general,

$$x \xrightarrow{f} x^3 + 5.$$

In this particular case, there is another function g, usually called the inverse function of f, defined by

$$g(x) = (x - 5)^{1/3}.$$

Then

$$13 \xrightarrow{g} 2,$$

$$-22 \xrightarrow{g} -3,$$

$$5 \xrightarrow{g} 0,$$

$$x \xrightarrow{g} (x - 5)^{1/3}.$$

Thus g is the function which undoes the work of f; clearly

$$f[g(x)] = g[f(x)] = x.$$

Quite often we like to think of a correspondence in a slightly more active manner [like a meat-grinder!]. We then usually speak of a function f as an operator which operates on the number x to produce the number f(x). Thus the rule

$$y = x^2 - 2x + 7$$

gives us an operator f; the operator f acts on 10 to produce f(10) = 87 [f operating on 10 gives 87]. In general, f operates on x to give $x^2 - 2x + 7$, that is,

$$f(x) = x^2 - 2x + 7.$$

This is very close to the "rule" definition of a function.

6. The Function f(x).

We frequently see "the function f(x)" as a phrase in books. Strictly speaking, we should always write "the function f whose value at x is given by f(x)", but this circumlocution would be unbearably pedantic. Consequently,we introduce a convention [we again quote from Serge Lang, A First Course in Calculus]. "In order to describe a function, we need simply to give its value at any number x. Thus we usually speak of a function f(x), meaning by that the function f whose value at x is f(x). For instance, we would say 'Let f(x) be the function $x^3 + 5$'."

Students can learn a great deal about functions by looking at particular examples which exhibit peculiarities. A number of less usual functions can be found in Chapter 7 of the book Topics in Modern Mathematics referred to in Section 1.

MODERN NON-MATHEMATICS BASED ON SETS

[Reprinted by permission of the Canadian Mathematical Bulletin and the author, Professor R.A.Staal.]

The great increase in population in the lower age groups in recent years has created a vast market for education, and this market has, in the way of all markets, created an interest in the production of material - in this case educational material. It is truly astonishing how many people who, a few years ago, would have considered the writing of introductory works on Mathematics to be either above them or beneath them, have been stimulated by such circumstances into acquiring an interest in what has been going on in Mathematics in the past 100 years, with a view to bringing Mathematics instruction up to date.

That these writers have approximated to the fundamental truths can be readily seen in the fact that their separate meditations have led to almost identical results, and all in the period of a few years. This applies not only to subject matter, but to terminology, and such things as the choice of illustrations (pictures of computing machines, etc.).

Now, this renaissance has proceeded at a rate which has, unfortunately, left most of the older (30 years or over) practising mathematicians far behind. This is a conclusion which has been reached, reluctantly, on searching the better-known mathematical journals for examples of modern usage, and finding it to be lamentably lacking, except in certain rather special fields. (Departments of Education have seemingly been more progressive in this regard.) Since the older folk are more likely to be stubbornly set in their ways, the great, surging, forward movement is appropriately being centred on our children, who are more likely to be receptive to what is placed before them.

In the case of our own department, we must confess that our record is no better than others. Even such (recently) well-established terms as "open sentences", "truth-set", "solution-set", "graph of a generalization", and "set-maker operation" seem to be totally unfamiliar, although a few moments of patient explanation seems to convey their sense. It is therefore partly toward our own department that this bit of exposition is directed.

To facilitate discussion, we use the following definition, intended to help in distinguishing the "old" Mathematics from the "new".

1.1 DEF non-(traditional) Mathematics $\longrightarrow$

> the intersection of 93% of the introductory Mathematics texts published in North America during the past 27 months

This will usually be abbreviated as "non-Math". (An imperceptive reader, of course, might be mislead through ordinary usage to a misinterpretation of this term. This is unfortunate, but I am of course using a rigorous definition, which should remove any such misconceptions.)

One characteristic of non-Math is that it is BASED ON SETS. Mathematics (i.e. non-non-Math, see 1.1) can also be based on sets, from one point of view, but it must be made clear that what our renaissance involves is the basing of non-Math on sets. This is what is really new, and a similar remark applies to other features.

Another characteristic is its appreciation of the importance of vocabulary and technical terminology. Most work on the foundations of Mathematics has been written in the language of set theory, and it therefore follows that as much as possible of the terminology used by workers in this field (Russell, Godel, etc.) should be introduced as early as possible in the schools. Indeed, there is an obvious correlation between the use of certain words and non-Math content.

As an example of the virtues of non-Math, let us see how a statement such as

2 is a root of (i.e. satisfies) $x^2 + 3 = 7$

which one frequently sees in Mathematics books becomes illuminated when it is analysed by set-theoretical means, thereby making its true meaning clear to the pupil.

1) We need the concept of an open sentence. This is not a sentence (in the usual sense of the word) which is "open" in any sense. It is, rather, what used to be called a propositional function (or form) and contains variables, the replacement of which by names of certain things makes the result a sentence. e.g., $x^2 + 3 = 7$.

2) Next we need the set-maker operation $\{\ |\ \}$. When the space to the right of the bar is filled with an open sentence (see 1)), and the space to the left with its variable, the result is the set of all values of the variable which, when their names are substituted for the variable in the open sentence, produce a true sentence (in the ordinary sense of the word). Everybody knows what a set is. e.g. a heap of stones. An example of the use of this operation is $\{x|x^2 + 3 = 7\}$ which is the set of all values of x which satisfy $x^2 + 3 = 7$.

3) Next we need the concept of a solution set. This is the set of all values of a variable which give a solution of an open sentence. i.e. the names of which, when substituted for the variable in the open sentence make it a true sentence in the ordinary sense.

e.g. $\{x|x^2 + 3 = 7\}$ is the solution set of the open sentence $x^2 + 3 = 7$

4) After this we need only the membership relation "ϵ". This is the relation which every element of a set bears to that set. (In a more thorough presentation of non-Math this relation would of course be defined as the set, R, of all ordered pairs $\langle x, S \rangle$ where x is an element of the set S. ~~Thus, $x \epsilon S$ simply means that $\langle x,S\rangle \epsilon \{y|(\exists x)(\exists S)(y = \langle x,S\rangle.)$~~

5) Finally we see that the sentence "2 satisfies $x^2 + 3 = 7$", by 1), 2), 3), 4) amounts simply to

$$" 2 \in \{x | x^2 + 3 = 7\}",$$

its true meaning thereby being made apparent. ~~(It amounts to this: 2 is an element of the set such that x is an element of the set if and only if x satisfies the equation x^2)~~

*(Please note: the above lines which were crossed out by the author were typed in by mistake. Sorry. Typist.)

Above all, non-Math concentrates on fundamental ideas. Since these are the building-blocks of non-Math (and Mathematics too) it follows of course that non-Math instruction should begin with a thorough discussion of them. This means, among other things, that distracting influences such as many of the problems found in the older books should be avoided, and replaced where possible by carefully graded exercises involving just the fundamentals themselves.

e.g. 1. What is the solution set of $x = 7$?

2. What is the solution set of $x > 7$?

3. What is the solution set of $x \geq 7$?

The principle of Maximum Exposure should be followed, for the greater the space and time devoted to an idea, the more deeply will it sink.

The question of how early non-Math may be begun has been receiving much attention, and some interesting discoveries have been made. While the following may be extreme at this time, it at least indicates the direction which experimenters are following.

"Oh see. Johny has a set of marbles. See Johny's set. Look, look, Billy has a set of marbles. See Billy's set. Here comes Mary. Mary gets all the marbles. Mary gets the union of Johny's set and Billy's set. See Mary's union."

While some may balk at such a radical venture, it has for a long time been the writer's view that very little of the content of non-Math is, when suitably presented, beyond the capabilities of a bright six-year old, and that a successful capitalization on this fact might well be a major step forward in non-Math education. It is perhaps this which makes non-Math so interesting to many educators.

Finally, the progressive character of non-Math shows up in the way it reflects modern technological developments. The conspicuousness of computers and the fact that their calculations are performed essentially to base 2, has brought to light the fundamental importance of scales of notation in non-Math, and has stimulated a great new interest in this topic, which in many of the newer non-Algebra texts is placed at the very threshold of the subject.

[continued on page 29]

REPORT FROM

THE ONTARIO MATHEMATICS COMMISSION

The Ontario Mathematics Commission held its Annual Meeting on Saturday, September 7, in Room 619 of the Toronto Education Centre. The entire minutes of the Commission Meeting cover twenty-eight pages; it is hoped that the following report can indicate some of the high points of the meeting. It is not meant to be exhaustive!

1. Resignation of Mr. Asbury.

The Secretary-Treasurer of the Ontario Mathematics Commission, Mr. Frank Asbury, submitted a letter of resignation at this meeting, his resignation to take effect not later than December 31, 1963. The President, Dr. Elliott, commented that the position would not be an easy one to fill, as Mr. Asbury had been of invaluable help to the Commission and to the President. The members of the Commission responded with a spontaneous tribute to Mr. Asbury and his work for the Commission. [Editor's Note: Mr. Henry Campbell has since accepted the position of Secretary-Treasurer of the Commission.]

2. Election of New Executive.

A nominating committee was appointed, headed by Professor DeLury. The nominating committee recommended that, in view of the fact that this was a very special year in which the Commission would have to deal with the Curriculum Institute and meet the particular difficulties arising from Mr. Asbury's resignation, there be no change in the membership of the Executive for the coming year. This motion was carried, leaving the Executive as follows:

Past President	-	Dr. John Coleman,
President	-	Dr. H. A. Elliott,
Vice-President	-	Mr. Norman Sharp,
Secretary-Treasurer	-	Mr. F. C. Asbury,
Executive Members	-	Dr. P. R. Beesack, Rev. J. C. Egsgard, Mr. J. C. Gardner, Mr. J. F. Kinlin.

3. Report of Scholarship Committee.

This report was submitted by Mr. Totton, Chairman of the Scholarship Committee. One of the high points was a report on circulation of a questionnaire dealing with the formation of mathematics clubs in secondary schools. A great deal of interest had been shown in this project and it was found that teachers wanted to get information about setting up clubs, suggestions for topics, suggested forms and books for the students. Lists of films, a bibliography of recommended texts of interest, and a list of possible topics were circulated to the schools. It was suggested

that each school have liaison with a nearby University. In this connection, Mr. Asbury wrote to each University asking that consideration be given to the appointing of some representative who would be willing to work with high schools in the area. At the date of the meeting, favourable replies had been received from Assumption, Carleton, Ottawa, Queen's, Toronto, Waterloo, and York.

4. Report of Committee on Elementary Education.

This report was presented by the Chairman of the Committee, Mr. Ryan. Among other work of the Committee, reviews of current experimental work and textbook development in elementary school mathematics had been made; consideration had been given to formulating a desirable outline and grade placement of topics in elementary mathematics; steps were taken to promote the circulation of copies of a brochure on "The Real Number System"; representations had been made to the Department of Education concerning summer courses for the benefit of teachers of intermediate mathematics. Discussion of the Cuisenaire movement had led to agreement that the movement had produced a desirable return to the effective use of concrete material and a realization of the ability of quite young pupils to go much further in arithmetic than the present course of study contemplates.

5. Report of the Committee on Secondary Education.

This report was presented by the Chairman, Dr. Beesack. The main concern of the Committee had been the prosed "Course A" for Grade XIII, a copy of which had been mailed to the Commission members. Dr. Beesack reminded the meeting that Course A is the proposed course for all students in Grade XIII who take any mathematics at all. A lengthy discussion followed of the topics in the proposed Course A, namely: sequences and series; mathematics of investment; introduction to logic; elementary set theory; permutations, combinations, and the binomial expansion; descriptive statistics; probability and statistics. The majority opinion seemed to be that many of the topics were too difficult for Grade XIII at the present time, and were far too difficult in notation and emphasis. Questions were also raised about the possibility of covering topics such as mathematical induction in the very limited time allotted to them. It was felt that the amount of set theory and symbolic logic introduced was more than could be covered in the time allotted. Questions on order of presentation were also raised. The main points of concern with most members of the Commission seemed to be the sheer volume of material and the level of difficulty of a number of the topics; it was finally, after an extended discussion, moved "That the Course A outline be referred back to the Secondary Education Committee for further study in the light of the discussion". It was also decided that the material should not be made available for teachers for trial until further study had been made.

6. Possible Amalgamation with the Ontario Curriculum Institute.

The Ontario Curriculum Institute, which was set up last November,

has plans for studying curriculum in the Province of Ontario. It has approached the Ontario Mathematics Commission, suggesting that the Commission become a part of the Ontario Curriculum Institute. With a view of investigating this possibility, three members of the Board of Governors of the Ontario Curriculum Institute, Rev. J. H. Conway, Dr. R. W. B. Jackson, and Mr. J. H. Mitchell, met for a considerable period of the afternoon with members of the Commission. Some of the points of the discussion are as follows.

Dr. Jackson explained that the concept of the Institute was as wide as the programme from kindergarten through to the end of university, and encompassed all subjects and aspects of the curriculum. The Institute was well aware of the work of the Ontario Mathematics Commission and had no intention of attempting to duplicate what the Commission had done. However, the terms of reference of the Institute required that the whole scope of the curriculum be covered and this most certainly included the subject of mathematics. Dr. Jackson stated that the question is basically one of the relationship between the Commission and the Institute.

Father Conway stated that it was the O.T.F's hope that the Institute would be able to get the same sort of results in other subject fields as had been realized by the Commission in the field of mathematics. Experts could be brought in to do research in different fields, thus enabling the Institute to go much further in curriculum study.

Various members of the Commission suggested points that needed to be discussed. It was pointed out that the work of the Commission had extended beyond the curriculum as such, and it was asked whether the operation of other activities of the Commission, such as the Mathematics Gazette, would be curtailed to any extent under the terms of reference of the Institute. It was asked whether the Commission could retain its present constitution if it merged with the Institute; to this Dr. Jackson replied that he did not know the answer.

It was pointed out that the Commission had eight different appointing bodies in addition to the universities, and it was asked whether this could continue or whether the Institute would be the sole appointing body. It was felt that the most contentious area of discussion could be the matter of the appointment of members to the Mathematics Commission. The Commission would certainly want to protect the present situation where, after quite a history of development, it had come to the stage where it was now a democratic body. It was doubted that the Commission would want to become a committee where the Board of Governors of the Curriculum Institute specified the membership. Clarification was asked for regarding the membership in the Institute, the Board of Governors, the number of members who are practising educators and the number who were administrators. Concern was also expressed about the possibility of the Board of Governors rejecting a report from a study committee.

One of the most fruitful lines of discussion arose from a question as to whether there was anything in the Constitution of the

Institute which would prevent the Commission from affiliating with the Institute without losing its Constitution or its name. It was also asked whether the Commission might have representation on the Board of Governors. Dr. Jackson replied that, at the present time, there was no provision at all for affiliation. He pointed out however, that at present the Institute was operating only with a preliminary charter. Regarding representation from the Commission and the Board of Governors, application would have to be made as with any other group. Some members of the Commission felt that a way could be found, perhaps by amendment of the constitution of the Curriculum Institute, for the Commission to affiliate with the Institute; apparently the Institute had no provision either for or against affiliation.

After the members of the Curriculum Institute had left the meeting, discussion continued, and a motion was made to the effect "That four members of the Executive constitute a committee to confer with a similar group from the Curriculum Institute as to possible terms of amalgamation and report back to the next General Meeting of the Commission".

In discussion it was pointed out that if the Commission were to become totally financially dependent on the Institute, the prospect for the Commission might be glum. Hence it was very important in the present negotiations to spell out any conditions very carefully.

It was suggested that possibly, if the negotiating committee found that there were very definite reasons for not going into the Curriculum Institute, and work for the Commission to do outside the programme of the Institute, the Commission might be in a position to go to the Department of Education and ask for financial aid. The question of affiliation with the present Commission constitution was again brought up.

The motion was finally passed with an amendment to the effect that the sub-committee should be "with powers to co-opt additional members of the Commission if so desired".

7. This summary has only been a summary, but it is hoped that it has, in some small measure, managed to convey a flavour of the proceedings of what has undoubtedly been an extremely important meeting of the Ontario Mathematics Commission.

Some Diophantine Problems - Part 1

[Being a version of a talk given at the Toronto Mathematics seminar, January 15, 1964, by R. G. Stanton]

1. Definition.

A Diophantine equation is one in which we seek solutions restricted to being integers. Sometimes we even restrict our solutions to be positive integers.

For example, consider the equation

$$2x^2 + y^2 = 43.$$

As a Diophantine equation, this is soluble. There are two variables x and y, but they can only assume integral values. Indeed, we can argue:

$$y^2 \geq 0; \quad \text{hence} \quad 2x^2 \leq 43; \quad x^2 \leq 21.5.$$

Thus $x = 0, \pm 1, \pm 2, \pm 3, \pm 4$, are the only possibilities. By the METHOD OF TRIAL, we find that $x = \pm 3$ is the only value of x that works in the original equation; then $y = \pm 5$ and the four solutions are $(3,5), (3,-5), (-3,5), (-3,-5)$.

Note that we can consider this problem geometrically; $2x^2 + y^2 = 43$ represents an ellipse in the xy-plane. Infinitely many points (x,y) lie on this ellipse, but the Diophantine problem is to find the points with integral co-ordinates (or lattice points) which lie on the ellipse.

2. Types of Diophantine Equations.

We can use the usual terminology (remember that all symbols in this article refer to integers unless otherwise stated). Thus

$$2x + 3y = 19$$

is a linear Diophantine equation in two variables;

$$\begin{aligned} 2x - y + z &= 17 \\ x + 3y - 4z &= 15 \end{aligned}$$

is a pair of linear Diophantine equations in three variables;

$$x^3 + y^3 = z^2$$

is a cubic Diophantine equation in three variables. One classification that is very useful is by types of solution. Thus we can speak of:

Type 1 Equations: No solutions exist;

Type 2 Equations: A finite number of solutions exists;

Type 3 Equations: Infinitely many solutions exist.

Note that we can not look at an equation, classify it as Type 1 or Type 2 or Type 3, and then plug in a method of solution! We have to find the solution first and then classify the equation or equations.

3. Equations with No Solutions.

We proceed to give three examples.

Example 4.1 $\quad 6x + 15y = 49$

We write $3(2x + 5y) = 49$.

If an integral solution (x,y) exists, then $2x + 5y$ is an integer, this equation states that 3 times an integer is 49. But this is impossible; so there are no integers x and y satisfying the given equation.

Example 4.2 Find all lattice points on the hyperbola

$$x^2 - y^2 = 506.$$

The case method used here is very potent. We see that there are four possibilities.

Case	1	2	3	4
x	odd	even	odd	even
y	odd	odd	even	even

In Case 2 and Case 3, $x^2 - y^2$ is the difference between an odd integer and an even integer; this difference must be odd and so can not equal 506.

In Case 4, we let $x = 2r$, $y = 2s$. Then the equation becomes

$$4(r^2 - s^2) = 506,$$

which is impossible since 4 is not a divisor of 506.

Finally, in Case 1, we let $x = 2p + 1$, $y = 2q + 1$. The equation becomes

$$4(p^2 + p - q^2 - q) = 506,$$

which again is impossible.

Thus we see that there are no solutions.

Example 4.3

Solve the Diophantine equation $x^2 = 8y + 103$. We see that this solution is equivalent to finding all lattice points on the vertical parabola $y = \frac{x^2}{8} - \frac{103}{8}$.

We divide into cases as in the preceding example and consider $x^2 - 8y = 103$. In both Case 2 and Case 4, x is even; so $x^2 - 8y$ is even and can not equal 103. So these case are impossible.

In Cases 1 and 3, x is odd and we may write $x = 2k + 1$. Then

$$4k^2 + 4k + 1 - 8y = 103,$$

$$4(k^2 + k - 2y) = 102.$$

But this is impossible, since 4 does not divide 102. So the given equation has no solutions.

4. Equations with a Finite Number of Solutions.

We have already given an example of such an equation in Section 1. We now take a rather interesting example.

Example 5.1

$$x + y + z = x^3 + y^3 + z^3 = 3.$$

We know a relationship between $x + y + z$ and $x^3 + y^3 + z^3$, namely,

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Substitute and divide by 3.

$$1 - xyz = x^2 + y^2 + z^2 - xy - xz - yz$$
$$= (x + y + z)^2 - 3(xy + yz + zx).$$

Hence $xyz - 3(xy + yz + zx) = -8$.

The left-hand side is the first two terms in the product of $(x-3)(y-3)(z-3)$; hence $(x-3)(y-3)(z-3) - 9(x+y+z) + 27 = -8$, that is $(x-3)(y-3)(z-3) = -8$.

Thus $x-3$, $y-3$, $z-3$, are three integers whose product is -8 (we disregard order for the moment). The only possibilities are

$-2, -2, -2$ or $-2, 2, 2,$ or $1, 2, -4$

or $1, -2, 4$ or $-1, 2, 4$ or $-1, -2, -4$

or $1, -1, 8$ or $1, 1, -8$ or $-1, -1, -8$

where there is either one negative integer or three. But not all these sets work; indeed

$$(x-3) + (y-3) + (z-3) = x + y + z - 9 = -6.$$

So we must seek a set whose sum is -6. The only possibilities are $(-2, -2, -2)$ and $(1, 1, -8)$. Trying

$$x-3 = y-3 = z-3 = -2,$$

we get $x = y = z = 1$, which clearly is a solution. Trying $x-3 = y-3 = 1$, $z-3 = -8$, we get $x = y = 4$, $z = -5$; this solution also checks in the original equations.

By rearrangement, this last solution gives two others symmetrically, namely, $x = z = 4$, $y = -5$; and $y = z = 4$, $x = -5$. So there is a total of four solutions.

5. Equations with Infinitely Many Solutions.

As an example of an equation with an infinitude of solutions, we select the famous Pythagorean equation $x^2 + y^2 = z^2$. Since any solution remains a solution by changing the sign of any one of x, y, z; and since any solution with a zero value is trivial; we

see there is no loss in generality by taking x, y, and z as positive integers.

Now any Pythagorean triplet, such as $(3,4,5)$, generates an infinity of solutions by magnification to give $(3K,4K,5K)$. We shall use the term primitive solution for a solution (x,y,z) where x, y, and z have no common factor. The relation $x^2 + y^2 = z^2$ shows that for a primitive solution, x and y have no common factor; similarly, x and z have no common factor; y and z have no common factor. We agree to search only for primitive solutions.

Now x and y can not both be even; for then z must be even and the solution is not primitive.

Also x and y can not both be odd; for then z is even and we get

$$(2r + 1)^2 + (2s + 1)^2 = (2t)^2,$$

which leads to the impossible result

$$4(r^2 + r + s^2 + s - t^2) = -2.$$

However, 4 is not a divisor of -2.

Thus we are led to the conclusion that x and y are such that one is odd, the other even. There is no loss of generality in assuming x even. So we can write

$$x^2 = z^2 - y^2 = (z-y)(z+y),$$

where x is even, y and z are odd.

Set $x = 2a$, $z - y = 2b$, $z + y = 2c$.

Then $a^2 = bc$.

Furthermore, since $z = c + b$, $y = c - b$, we see that any common factor of b and c also divides y and z. But y and z are relatively prime (have no common factor). Thus b and c have no common factor.

Since $a^2 = bc$, we see that b and c are each perfect squares (b and c have no common factor). Thus

$$b = m^2, \quad c = n^2, \quad a = mn.$$

This at once gives the solution:

$$x = 2mn, \quad y = n^2 - m^2, \quad z = n^2 + m^2$$

where m and n are relatively prime (note that both of m and n can not be odd in order to avoid a common factor in y and z).

The above solution is primitive. As an illustration, $n = 5$, $m = 2$, gives the Pythagorean triplet 20, 21, 29. The general non-primitive solution is simply:

$$x = K(2mn), \quad y = K(n^2 - m^2), \quad z = K(n^2 + m^2),$$

where m and n are relatively prime, one being even, the other odd.

THAT REMARKABLE NUMBER

Ross Honsberger (Northview Heights)

This is an account of an aside, inserted in a course of lectures in Number Theory given in the Fall of 1963 at the University of Waterloo, by the British mathematician, Professor W. B. Pennington.

In the theory of numbers, there is a function, $r(n)$, which is defined as the number of ways an integer n can be expressed as the sum of two squares of integers. Different ordered pairs of such integers are counted separately; for example, $r(5) = 8$ because

$$5 = (+1)^2 + (+2)^2 = (+2)^2 + (+1)^2 = (-1)^2 + (+2)^2 = (+2)^2 + (-1)^2$$
$$= (+1)^2 + (-2)^2 = (-2)^2 + (+1)^2 = (-1)^2 + (-2)^2 = (-2)^2 + (-1)^2,$$

corresponding, respectively, to the ordered pairs

$$(1,2),\ (2,1),\ (-1,2),\ (2,-1),\ (1,-2),\ (-2,1),\ (-1,-2),\ (-2,-1).$$

A few values of this function are

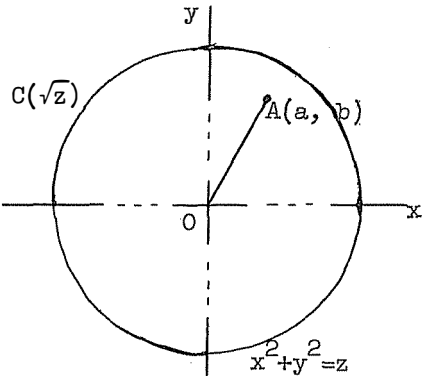
$$r(0)=1,\ r(1)=4,\ r(2)=4,\ r(3)=0,\ r(4)=4,\ r(5)=8,\ r(7)=0,\ r(12)=0.$$

There are various little facts that can be proven about $r(n)$; for example, $r(n) = 0$ if n is a prime number whose remainder is 3 when divided by 4 (i.e., if the prime $n \equiv 3 \pmod 4$; and there is an endless sequence of such primes. Thus every now and again the value of the function drops to zero as n runs through the integers $0,1,2,3, \ldots$. Sometimes $r(n)$ gets very large, a hint of which is given by $r(1105)=24$. It is a very irregular function; the functional values even for adjacent integers may differ greatly. No formula is known for it. Accordingly, without a formula to supply us quickly with individual values of $r(n)$, it occurs to us to take a look at the average value of the function over an interval. And this is where the function $r(n)$ becomes surprisingly interesting.

The average value of $r(n)$ over the integers $0,1,2,3,\ldots,(z-1)$ is $[r(0) + r(1) + r(2) + \ldots + r(z-1)]/z$. If we call the numerator by the name $R(z)$, we may denote this average more briefly by $R(z)/z$. The average over the entire range of non-negative integers is the limit of this expression as z increases indefinitely. While both $R(z)$ and z become indefinitely large as z increases without limit, the amazing thing is that the value of the limit is that remarkable number π ! On the average, then, a non-negative integer has π ordered pairs giving rise to expressions of it as the sum of the squares of two integers.

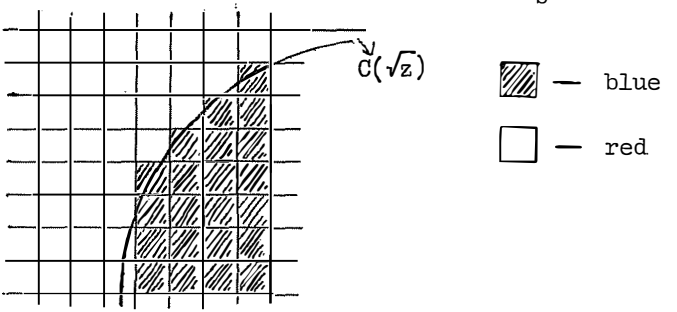
One might wonder how the number π could have any relation to a thing like $r(n)$. And it would be expected that the proof of the result would require a sophisticated and subtle approach. However, the proof is not difficult. It is highly instructive and very interesting. (It was proved by Gauss around 1800, at age 23.)

Proof. In the Cartesian co-ordinate plane, consider the circle with centre at the origin and radius $\sqrt{z}$. It has equation $x^2 + y^2 = z$ and is denoted by $C(\sqrt{z})$. Inside this circle are some lattice points, i.e., points whose co-ordinates are integers. Let $A(a,b)$ be any lattice point inside $C(\sqrt{z})$. Then OA, O the origin, is less



than $\sqrt{z}$, the radius. That is, $\sqrt{a^2 + b^2} < \sqrt{z}$, or $a^2 + b^2 < z$. Thus for some value n less than z, the ordered pair (a,b) leads to an expression of it as the sum of the squares of two integers; $a^2 + b^2 = n < z$. Every lattice point in $C(\sqrt{z})$ contributes 1 to the value of $R(z) = r(0) + r(1) + \ldots + r(z-1)$, since it is in the set of ordered pairs belonging to some r(n) in R(z). Conversely, any ordered pair (p,q), such that $p^2 + q^2 = n < z$, i.e., any ordered pair belonging to R(z), must be the co-ordinates of a lattice point in $C(\sqrt{z})$. (The distance from (p,q) to 0 is less than the radius $\sqrt{z}$ since $p^2 + q^2 < z$.) Consequently, R(z) is exactly equal to the number of lattice points in $C(\sqrt{z})$.

Around each lattice point in the plane, put a unit square such that the lattice point is at the centre of the square. Colour blue the squares around the lattice points inside $C(\sqrt{z})$ and red all other squares. Most of the circle $C(\sqrt{z})$ is blue and most of the outside of $C(\sqrt{z})$ is red. Some blue squares may project beyond $C(\sqrt{z})$ and some red ones cut into $C(\sqrt{z})$. The number of blue squares is the number of lattice points in $C(\sqrt{z})$ and that is the value of R(z). Since the squares are unit squares, the number of blue squares is the numerical value of the area coloured blue, A_b. Thus, $R(z) = A_b$.



The length of the diagonal of a unit square is $\sqrt{2}$. One-half of the diagonal has length $1/\sqrt{2}$. Since the centres of red squares are on or outside $C(\sqrt{z})$, the farthest a red point could project into $C(\sqrt{z})$ is $1/\sqrt{2}$, one-half the length of the diagonal. For it to be this far in, the centre of its square would have to lie on $C(\sqrt{z})$ and the diagonal of the square would have to be in line with the radius of $C(\sqrt{z})$ to its centre. Thus no red point is far enough inside $C(\sqrt{z})$ to be inside the circle $C(\sqrt{z} - 1/\sqrt{2})$, with centre 0 and radius $\sqrt{z} - 1/\sqrt{2}$; that is, this circle covers only blue points. Thus A_b must be at least as big as the area of $C(\sqrt{z} - 1/\sqrt{2})$.

By similar reasoning, we see that blue points can not project beyond $C(\sqrt{z})$ even as far as $1/\sqrt{2}$ because the centres of blue

squares are strictly inside $C(\sqrt{z})$. Consequently, the circle $C(\sqrt{z} + 1/\sqrt{2})$ must contain all the blue points. There are no blue points outside it or on it. Thus A_b can not exceed the area of $C(\sqrt{z} + 1/\sqrt{2})$.

Combining these two results, we obtain

$$\text{area } C(\sqrt{z} - 1/\sqrt{2}) \leq A_b = R(z) \leq \text{area } C(\sqrt{z} + 1/\sqrt{2}),$$

$$\pi(\sqrt{z} - 1/\sqrt{2})^2 \leq R(z) \leq \pi(\sqrt{z} + 1/\sqrt{2})^2$$

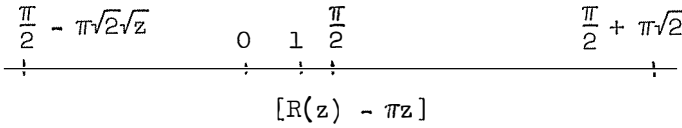
$$\pi(z - \sqrt{2}\sqrt{z} + 1/2) \leq R(z) \leq \pi(z + \sqrt{2}\sqrt{z} + 1/2),$$

$$\pi/2 - \pi\sqrt{2}\sqrt{z} + \pi z \leq R(z) \leq \pi/2 + \pi\sqrt{2}\sqrt{z} + \pi z.$$

Subtracting πz throughout, we get

$$\pi/2 - \pi\sqrt{2}\sqrt{z} \leq R(z) - \pi z \leq \pi/2 + \pi\sqrt{2}\sqrt{z}.$$

Thus the number $R(z) - \pi z$ is to be found somewhere in the interval from $\pi/2 - \pi\sqrt{2}\sqrt{z}$ to $\pi/2 + \pi\sqrt{2}\sqrt{z}$ (end points included).



Consequently, the magnitude of $R(z) - \pi z$ can not exceed $\pi/2 + \pi\sqrt{2}\sqrt{z}$.

Thus $|R(z) - \pi z| \leq \pi\sqrt{2}\sqrt{z} + \pi/2$

Now $\pi\sqrt{2} < 8$. Thus we may write $|R(z) - \pi z| < 8\sqrt{z} + \pi/2$.

Also, for $z \geq 4$, $\sqrt{z}$ is greater than $\pi/2$. Thus, for $z \geq 4$, $8\sqrt{z} + \sqrt{z} = 9\sqrt{z}$ is greater than $8\sqrt{z} + \pi/2$. For $z \geq 4$, then $|R(z) - \pi z| < 9\sqrt{z}$. Dividing throughout by z, we get

$$\left| \frac{R(z)}{z} - \pi \right| < \frac{9}{\sqrt{z}}.$$

This relationship holds for each "large" value of z as z increases without limit.

Clearly, as $z \longrightarrow \infty$, $9/\sqrt{z} \longrightarrow 0$. Thus, as $z \longrightarrow \infty$,

$\left| \frac{R(z)}{z} - \pi \right| \longrightarrow 0$, giving $\lim_{z \to \infty} \frac{R(z)}{z} = \pi$.