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ONTARIO MATHEMATICS GAZETTE

E D I T O R I A L S T A F F

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E D I T O R I A L

Well, here we are back with another Gazette after the summer break. I hope that all readers had a pleasant summer and will not be too critical of an editorial made up of a number of disjointed remarks.

One of the features of this Gazette I should like to call to your attention is the article by Bill Nediger on the Junior Mathematics contest in District 10. Bill and the other men running the contest did a great deal of work on it and deserve a great deal of credit; the contest appears worthy of spreading to other districts and I hope that many more schools will take part in the future.

The Gazette has also received the compliment of having one of its articles reprinted. The "Wisconsin Teacher of Mathematics" (June, 1963) reprinted the article on "Sets and Functions" by Professor R. A. Staal.

During the summer, I had the opportunity to spend a month in Saskatoon giving a series of lectures at the Canadian Mathematical Congress. While there, I had the good fortune to have several long discussions with Saskatchewan educators and I would like to pass on to readers of the Gazette some of the opinions of Professor A., who also knows the Ontario set-up very well.

I had been commenting on the fact that here, in Ontario universities, we found that students with a senior matriculation (and good marks) from Saskatchewan (Grade 12) did very well when they enrolled in Arts courses but were generally not very successful when they enrolled in Science courses. Professor A. gave the opinion that students from Saskatchewan interested in science should remain at home where the courses took their background into account. He pointed out that Saskatchewan Grade 12 was equivalent to no more than $12\frac{1}{2}$ years of the Ontario programme at the most, and that consequently Saskatchewan students entering an Ontario university were at least half a year behind in their work. This was of no importance in subjects such as psychology, economics, philosophy, etc., which are begun in university. It was an enormous handicap to students in mathematics and science, where the University builds up the work previously taken.

Certainly my own experience supports the comments made by Professor A. I have found students from Saskatchewan, and some other provinces with 12 year programmes, very badly prepared in Mathematics. This is a point to bear in mind here in Ontario, where numerous politicians seem very free with grandiose projects for educational reform. Improvement of education in Ontario is a matter which concerns us all; but it is disheartening to see easy recipes for improved education springing up overnight in political platforms. How much thought has gone into promises to abolish Grade 13? How much knowledge of education lies behind suggestions that admission to University be on the basis of 50% in 6 Grade 13 subjects? Does Ontario need a college in every town, as if a college were another "home-town industry" like a shoe factory?

Educators at all levels in the province need to keep a vigilant eye open against attempts to use education as a political football. Let us maintain proper standards; let us be wary of "plans" and "programmes" which are born in the heat of political battle.

I do not mean to suggest that the Ontario system is perfect at the moment; indeed, I have been very critical of some aspects in the past. Many suggestions are worthy of profound study. Perhaps it would be better if the dual-purpose aspect of the Grade 13 examinations was abolished, and if there was a set of provincial examinations for senior matriculation, another set of University exams for University entrance. But questions such as these should be settled after careful scrutiny by educators, not after hasty survey by politicians.

And now, au revoir for another month. If you are having the same problems with bulging enrolments that I am encountering, I wish you plenty of stamina for the next few months! All best wishes for a good year in 1963-64.

Ralph G. Stanton

Editor-in-Chief

[BAD FORM: continued from page 36]

4. CONFUSION OF THE TERMS EQUATION AND FUNCTION

Example 14. "The new quadratic equation is $3x^2 - 7x + 4$."

This statement can not be correct! an equation must contain an equals sign. A correct statement would be: "the new quadratic equation is $3x^2 - 7x + 4 = 0$".

Example 15. "The graph of the equation $x^2 + 2x + 5$ is a parabola."

Again $x^2 + 2x + 5$ can not be an equation; read: "the graph of the function $x^2 + 2x + 5$ is a parabola".

THE JOBLINIAN BOOK OF GEOMETRIC THEOREMS

by Fred Joblin

[Editor's note: The following article was submitted by M.A. Neuwelt of Port Credit Secondary School: it is the work of one of his students. We publish it as an excellent example of deductive reasoning with a bit of humour added. The reader will undoubtedly find spots where a more experienced student could have shortened the development; however, the significant thing is the pattern of mathematical thought proceeding from definitions to theorems thereon.]

Definitions

1. The joblection of a point is the convex figure formed when any arc is drawn from a given point, and another arc cutting the first arc in two places is drawn from any point on the first arc, having the same radius.

2. The hyrotuse points are the two points of intersection of the two arcs.

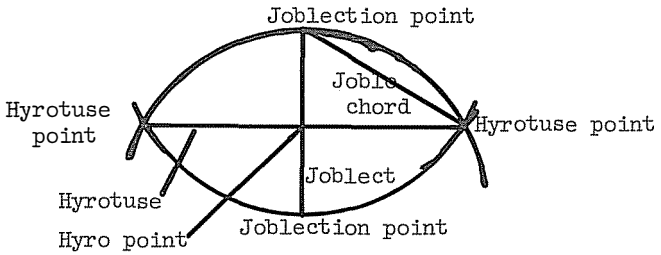
3. The hyrotuse is the line joining the two hyrotuse points.

4. The joblection points are the two points on the arcs from which the arcs were drawn.

5. The joblect is the line joining the two joblection points.

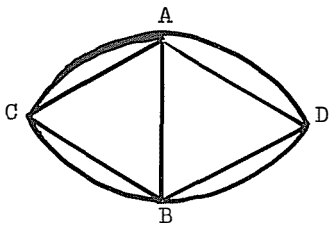
6. The hyro point is the point of intersection of the hyrotuse and the joblect.

7. A joblo chord is a line joining either hyrotuse point with either joblection point.



Proposition 1. A Theorem.

In a joblection, all joblo chords, and the joblect, are equal.



Given: A joblection with the joblect, AB, and all joblo chords AC, AD, BD, and BC.
Required: AB = AC = AD = BD = BC

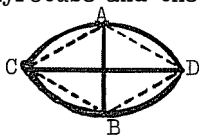
Proof: AC = AB = AD (equal radii)
BA = BC = BD (equal radii)

But AB = AB
$\therefore$ AB = AC = AD = BD = BC.

Corollary. Any lines, drawn from either joblection point, and terminating on the opposite side of the joblection, are equal.

Proposition 2. A Theorem.

In a joblection, the hyrotuse and the joblect right bisect each other.



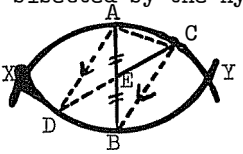
Given: A joblection, with the hyrotuse, CD, and the joblect, AB.
Required: AB and CD right bisect each other.

Proof: Join AC, AD, BC, and BD.
Then AC = AD = BC = BD ($\overline{V}$-1)
$\therefore$ figure ADBC is a rhombus
$\therefore$ AB and CD are diagonals of the rhombus ADBC,
$\therefore$ AB and CD right bisect each other (I-23) and (properties of a rhombus).

Corollary. In any parallelogram inscribed in the joblection with either the joblect or hyrotuse as a diagonal, the other diagonal of the parallelogram passes through the hyro point.

Proposition 3. A Theorem.

When a line is drawn through the hyro point from any point on the joblection and terminates on the opposite side of the joblection, that line is bisected by the hyro point.



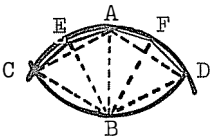
Given: A joblection with the joblect, AB, the hyro point E, and any point C on the arc XAY.
Required: CE, when joined and produced to meet the arc XBY, is bisected by the hyro point.

Proof: Take only CE. Join CB.
From A, construct AD||CB (I-15)
Join DE, AC, DB.
AD = CB ($\overline{V}$-1 Cor.) and AD||CB
$\therefore$ AC = DB and AC||DB (I-22)
$\therefore$ figure ACBD is a parallelogram.
$\therefore$ the diagonals bisect each other (I-23)

$\therefore$ E is the midpoint of AB; $\therefore$ E is a point on the other diagonal.
$\therefore$ DEC is a straight line, and DE = EC (I-23)

Proposition 4. A Theorem.

If in a joblection, two equal chords are drawn having either joblection point as a common vertex, then the lines joining the nearest hyrotuse point to each chord, and the point on the joblection where each chord meets the joblection, are equal.



Given: A joblection with chord AE = chord AF.
Required: CE = FD.

Proof: Join AB, CB, EB, FB, DB, CA, and AD.

In $\triangle$'s AEB, AFB, AB = AB
EB = BF ($\overline{V}$-1)
AE = AF
$\therefore$ $\triangle$ AEB $\equiv$ $\triangle$ AFB (I-4). $\therefore$ $\angle$ EBA = $\angle$ ABF

In $\triangle$'s ACB, ADB, AC = AD ($\overline{V}$-1)
AB = AB
CB = BD ($\overline{V}$-1)
$\therefore$ $\triangle$ ACB $\equiv$ $\triangle$ ADB (I-4). $\therefore$ $\angle$ CBA = $\angle$ DBA

$\therefore$ $\angle$ CBA - $\angle$ EBA = $\angle$ DBA - $\angle$ ABF (Axiom of subtraction)

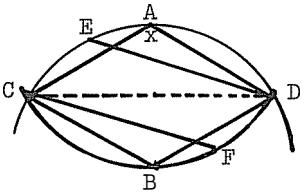
$\therefore$ $\angle$ EBC = $\angle$ DBF.

In $\triangle$'s EBC, FBD, CB = BD ($\overline{V}$-1)
BE = BF
$\angle$ EBC = $\angle$ DBF
$\therefore$ $\triangle$ ECB $\equiv$ $\triangle$ FDB (I-2)
$\therefore$ CE = FD.

Corollary. When two chords are drawn equal, one from either joblection point in a joblection, and on the same side of the joblection, then the lines joining the two points on the joblection of the chords to the joblect point on the same side of the chords, are equal. (Converse of $\overline{V}$-4)

Proposition 5. A Theorem.

Any lines forming any angles on the edge of the joblection, with the hyrotuse as the common base of the formed triangles, make the angles on the edge of the joblection all equal to 120°.



Given: Joblection with hyrotuse CD; angles at the centre: $\angle$ CAD and $\angle$ CBD. Any angles CED and CFD with vertices on the edge of the joblection.
Required: $\angle$ CAD = $\angle$ CED = $\angle$ CFD = $\angle$ CBD = 120°.

Proof: $\angle$ CED = $\frac{1}{2}\angle$ CAD (III-6)
$\angle$ CBD = $\frac{1}{2}\angle$ CFD (III-6)

Figure ADBC is a rhombus as AC = AD = BC = BD($\overline{V}$-1)

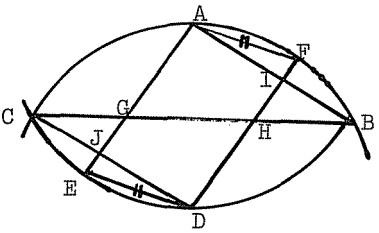
$\therefore \angle$ CAD = $\angle$ CBD (I-23)

$\therefore \angle$ CED = $\angle$ CAD = $\angle$ CBD = $\angle$ CFD.

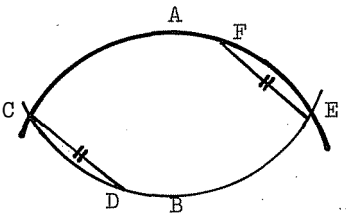
Let $\angle$ CAD be x.
Reflex $\angle$ CAD = 2 $\angle$ CBD, but $\angle$ CBD = $\angle$ CAD.
$\therefore$ Reflex $\angle$ CAD = 2 $\angle$ CAD
$\therefore$ Reflex $\angle$ CAD = 2x.
Reflex $\angle$ CAD + $\angle$ CAD = 360°
$\therefore$ 2x + x = 360°
$\therefore$ 3x = 360°
$\therefore$ x = 120°
$\therefore \angle$ CAD =(x) 120°
$\therefore \angle$ CED = $\angle$ CAD = $\angle$ CBD = $\angle$ CFD = 120°.

Corollary. In the rhombus formed by the joblo chords, or in any parallelogram with the hyrotuse as a diagonal, the angles at the hyrotuse points, are equal to 60°.

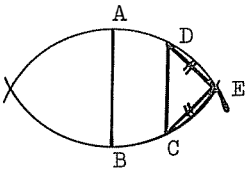
In the diagram, prove IH=GJ.



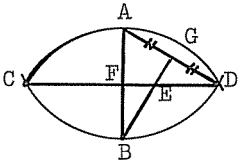
In the diagram, prove CD||EF.



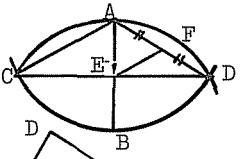
In the diagram, prove points A,B,C,D, are concyclic, and prove AB||DC



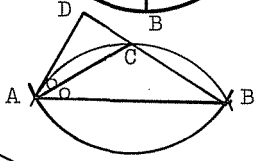
Prove BE=ED, and find the number of degrees in ∠ BED.



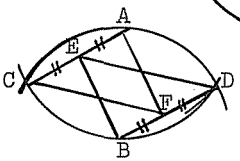
In the diagram, prove ∠ EFD =2 ∠ EAD, and that △ AEF is equilateral.



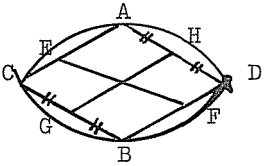
Prove D is a point on a circle having AB as diameter.



In the diagram, prove △BED ≡ △AFC

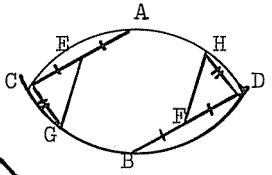


In the diagram, prove that any line EF, which is terminated by AC and BD, is bisected by GH.

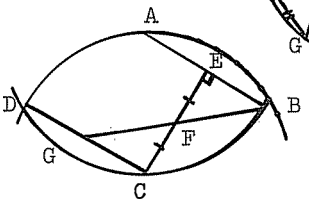


Construct a rectangle inside a joblection, with the joblect as a diagonal.

Using Problem 2 as an authority. Prove EG = HF, EG||HF.

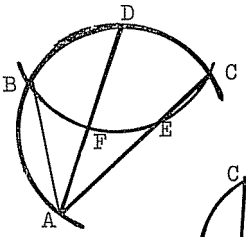


Prove: (i) BF = FG
(ii) DG = GC

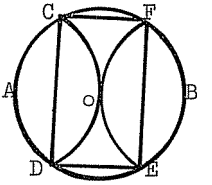


From either hyrotuse point of a joblection, draw any line passing through either curved surface, terminated by the circumference of the other arc produced. Joint the point on the circumference, and the point on the arc to the other hyrotuse point. Prove that the triangle so formed is equilateral.

Prove $\angle BAD = \angle DAC = 30^{\circ}$



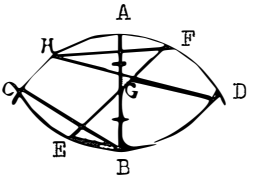
If AOB is a straight line, prove CDEF is a rectangle.



Show an alternate method for proving Proposition 8, Book 3 of Euclidean Geometry, by use of joblections.

Draw a line AC in a joblection from a joblection point A, terminated at the opposite side of the joblection, and passing through the joblo chord BD. Find the number of degrees in $\angle BCD$.

In the diagram, AG = GB, and H is any point on the arc CAF. Prove $\angle FHD = \angle CBE$.



PROPORTION PROBLEMS

H.H.Russell - Scarborough

The sequence of lessons presented in this paper have been designed for below-average high school students. There may be good reason to consider the material to be suitable as well for average classes at both the grade seven and eight levels.

Perhaps a brief review of some recent developments in psychological theory is in order at the outset, because the theory in question, that of Jean Piaget, seems to pertain directly to the curriculum material to be discussed. For the benefit of the newer members of the profession, Jean Piaget is an eminent Swiss psychologist from the Jean-Jacques Rouseau Institute who, for the past thirty years or more, has been experimenting with young children and adolescents in an effort to determine the growth patterns of some of man's most vital concepts and some of his basic patterns of thought. His work which is of most significance to mathematics teachers is contained in "The Growth of Logical Thinking from Childhood Through Adolescence" which he published in 1958 in collaboration with Mme.B.Inhelder.

The central idea in Piaget's tentative conclusions is that students below age 13 are unable to use simple basic strategies, i.e., logical operations, in everyday problem situations, whereas the students above the dividing line are able to use effectively, whether consciously or not, some of the basic strategies. One example may help to clarify to some extent what Piaget actually

did in his experiments, and furthermore it may help in the analysis of the sequence of lessons which follow.

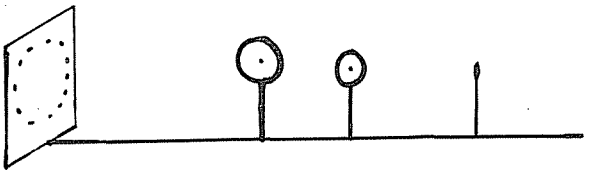


Figure 1.

Figure 1 shows one apparatus arrangement which Piaget uses. It consists of a source of light, a screen, and two mounted rings with different diameters. The student is asked to arrange the rings so that the shadows are the same size. After he has had time to experiment with the apparatus he is asked to explain it. The responses which Piaget records are extremely interesting, and they seem to fit well into a basic pattern which emerges from the data of several other tests of a similar nature. Piaget finds that students below age thirteen offer many interesting explanations, none of which is correct; whereas the older students nearly always offer an explanation which is essentially correct, that is, it involves the proportional relationship of the distances and the diameters. It is important to recognize that none of the students is likely to have learned about such relationships in regular classes. Hence Piaget concludes that a proportion "schema", or mental mechanism for handling proportions, becomes available to the students through maturation at about the age of fourteen.

It is probably true to state that most mental abilities grow or develop slowly through continuous increases; however, there now seems to be some evidence that certain abilities and skills have periods during which there is a potential for accelerated growth. It may be so with the logical ability required in the understanding of proportions and other structures which Piaget labels "formal". If this is in fact the case, then care should be taken to arrange curriculum materials so that students are not required to move through a sequence of lessons for which they are not yet sufficiently mature in the psychological sense. Failure to take such care may mean not only that material is out of place from the point of view of optimum scheduling, but also that the less mature students may be encouraged to adopt invalid substitute procedures for solving problems because the valid procedures are meaningless to them.

Some teachers may suspect that many of their own students in grades nine and ten fall into the low category described by Piaget, in spite of Piaget's claim that the dividing line is about age fourteen. The students in question may be fifteen or sixteen. This same suspicion prompted the writer to investigate the situation a little further and a report of this investigation, supporting the suspicion, may be found in "The Measurement

of Reasoning Ability in Adolescents" in the Autumn issue of the Ontario Journal of Educational Research.

One topic which seems to be out of place in the lower grades, or at least seems to be presented in an unnecessarily formal manner, is that of proportion problems. The sequence of lessons that follow was designed for use with a "prep" nine class, i.e., transfer nine, at Midland Avenue Secondary School. Besides the psychological advantages which will be obvious as the lessons progress, there are several other advantages to the sequence which will be discussed in the conclusion under the headings mathematical, and educational.

In the first lesson the situation shown in Figure 2 was discussed. The students were told that they could represent the rate, four books for forty cents, by the rate pair (4,40) and that they could refer to it as four per forty. As a check they were asked to use rate pairs to represent several other rates such as three books for thirty cents and five books for fifty cents. The students had no difficulty here and they seemed to be alert to the fact that order was important when they were asked about the separate parts of the rate pairs.

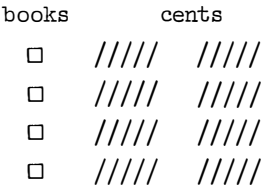


Figure 2.

The next step in the lesson was to develop and then consolidate the idea of equal (equivalent) rate pairs by intuition only, using the concrete cases as a basis for the intuitive judgements. The students did not need help when they were asked what they knew about the rate pairs (2,20), (3,30), (5,50) and (8,80). Also, it did not require much time for them to realize that (3,15) represented a different rate. At this point the students were not asked why there was a difference, but they were encouraged to find other rate pairs which were equal to those already considered, and they were asked to name other sets of rate pairs all of which were equal. Incidentally, the phrase "rate pair" was used from a very early point in the lesson, and prior to its use the phrase "number symbol" was used.

The next step was to have the students consider Figure 3, with a view to discovering the rules which govern equality of rate pairs. If the students are confronted with the equal rate pairs without the connecting bars they are likely to derive the relationship which they were given in grades six and seven, i.e., $3/30 = 4/40$. Because of the bars the students are led to another way, equally basic, which may be used to compare rate pairs. The next step was to have the students try other pairs known to be equal or unequal and thus consolidate the basic idea.

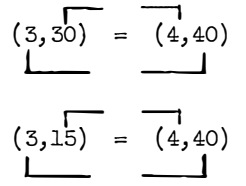


Figure 3.

The second lesson was to a large extent just a review of the first lesson, and it included considerable practice in comparing rate pairs. Also several word problems involving concrete examples of rate pairs were presented so that the students had an opportunity to translate some common words into the new symbols.

The third lesson really involved the kind of proportion problems which the students will meet later in science as well as in the higher grades in mathematics. First, the students were asked whether or not the rate pairs (3,4) and (6,8) were equivalent. Incidentally, the correct word was used here and from this point on, even although the symbol "=" was retained. This was not explained to the students, but in time, many of them had adopted the correct word and they seemed able to use the word "equal" when it was required.

In any event the students recognized that (3,4) and (6,8) were equivalent, and they knew the mechanical operations needed to show it.

Next, the students were asked whether or not the rate pairs (3,4) and (9,x) were equivalent. Even the slowest students -- and this takes the I.Q. down to about 70 -- seemed to have no difficulty either in realizing that some doubt existed as long as x was unknown or that a value for x of 12 would make the rate pairs equivalent. Next they were asked to try (5,8) and (10,x) and, surprisingly, nearly all of the students relied on the equation $5x=80$ to get their answer. The remainder of the period was devoted to solving other equations such as these, and several examples were done on the board to illustrate good form in the solutions. Among the examples were displayed solutions for each of the four possible positions of the x.

The fourth period was devoted to a review of the main ideas, especially that of order. Changing the order of one pair presented a minor challenge for some students, and in the case of changing the order of both pairs, most of the students predicted that there would be a corresponding change in the result. When they found that the change did not materialize they were especially anxious to examine the whole question of order more thoroughly.

The study of rate problems really began in period five. The initial example was "Seven ounces of chocolate cost nineteen cents. At the same rate, how much should fourteen ounces cost?" The students knew the answer from the beginning but they seemed to sense that this was a type of problem which had caused them trouble in the past and that perhaps their new rate pairs could help resolve some of the difficulty.

They knew enough to let x cents represent the cost, and they knew enough to get the numbers of ounces and cents in the proper places in the rate pairs. In fact, nearly all of the students were willing and able to state what should be written on the board next, at each step of the solution. After one other illustration the students were set to work on questions such as the following: "John earns $1.50 for every 90 papers he delivers. How many papers must he deliver in order to earn $5.00?" More

complex problems were presented on the second day, and by the third day the students were reasonably competent to begin almost any type of rate problem.

Although most of the problems which were studied in the last sequence of lessons were of the rate variety, there were, nevertheless, a few problems of the comparison type. This, naturally, led directly to the study of per cent, and as an introduction the following problem was presented to the students. "As the Winter term came to an end, John began to compare his test marks in his different subjects, 28 out of 40 in English, 33 out of 50 in Science, 42 out of 60 in History, and 64 out of 80 in Physical Education. How can John represent these marks with rate pairs? What rate pairs would result if John changed his scores to marks out of 100?"

The students had no difficulty performing the necessary calculations, and furthermore, they had no difficulty seeing the advantage of representing the marks out of 100. At this point the rate pair (75,100) was written on the board and the students were told that this is what they really mean when they write 75%. This revelation led to much activity converting per cents to rate pairs and vice versa, and after a full period of mechanical work the students were ready for per cent problems.

As far as the computation was concerned this was just a repetition of the rate problem lesson with the exception that there were only three places for the x rather than four as was the case with rate problems. Again the strategy in the presentation was to offer only one basic approach to the problems even although three distinct types were involved. Several different "type" solutions were completed on the board, and in the array were included examples of each of the three cases usually dealt with separately. The students had no difficulty at this stage, but at a later stage when they were required to deal with problems in which some manipulation was required in order to devise rate pairs, it was necessary to review and drill a little on the essential nature of the rate pair. This involved at one stage the use of such things as "(marks earned, marks possible)" just to illustrate what is needed in the first term and what is needed in the second term of each rate pair to be used.

The students completed the sequence of lessons on per cent, and they seemed to be confident of their most recent technique for handling per cent problems. Not one student asked if he could use the old way, and no one complained that he could not understand the problems.

An evaluation of these lessons may be made under many different headings. Only the headings mathematical, psychological, and educational are considered here.

From the mathematical point of view it is apparent that this sequence of lessons separates the frequently confused concepts of rate and comparison, and furthermore it establishes a symbolism which will make possible the separation of the concept of fraction-

al parts from both of them. Another mathematical feature is the informal introduction to equivalence relations and ordered pairs, which may be made formal without much difficulty. Also there is a very useful extension of the student's interpretation of algebra through his experience in the new notation. The fact that zeros are permitted in rate pairs, as in ratios, may be suggested as a mathematical advantage of the sequence of lessons, but perhaps it is equally important as an educational advantage.

Under the heading psychological advantages comes first the postponement of proportions in the formal classical sense. In reality, proportions have not been postponed, because the students master the techniques they need for solving proportion problems; however, the pitfalls usually associated with proportions seem to be missing. The other important psychological feature of the sequence of lessons is the reliance on simple concrete situations, such as "four notebooks cost forty cents", with the result that the students gain some insight into the basic relationships in a proportion through a knowledge of familiar rates and comparisons.

The educational features of the sequence of lessons include: first, the fact that the students not only accept the new way of doing things but also enjoy using the power it provides; secondly, the possible extension of the rate pair concept to suit the many situations in science where students are frustrated by the complexities of a proportion. A third feature may seem to be trivial, that of the minimal use of fractions and division, and yet it is founded on the traditional pedagogical idea that only one thing should be taught at a time. Undoubtedly fractions must be used correctly by the students, but it may be a good feature of these lessons that work with fractions comes later.

CORRESPONDENCE

Dr. Ralph G. Stanton,
Editor-in-Chief, Ontario Mathematics Gazette,
University of Waterloo, Waterloo, Ontario.

Dear Sir,

It might be interesting for the readers of the Gazette to see the matriculation examination questions of other countries.

Enclosed are the examination questions in the four branches of mathematics, set in 1962, for the Hogere Burger Scholen (5 year course) in Holland.

I would like to emphasize the fact that all students taking this course must matriculate in <u>all</u> fifteen subjects: four branches of mathematics, mechanics, physics, chemistry, biology, Dutch, English, French, German, history, geography, and political science.

Yours truly,

Karin B. Perkins.

MATH EXAM
HOLLAND 1962
IN GAZETTE OCT 1963

TRIGONOMETRY AND CALCULUS

1. (a) Given $\cos x = 3/5$. Find, without calculating x, the value of $9\tan x + 16\tan(x/2)$.

   (b) Solve for x: $4\sin^2 x + 6\sin x.\cos x = 4\cos^2 x - 1$.

   (c) Find the values of x for the interval $0 \leqq x \leqq 2\pi$ for which $\sqrt{2\cos x + 1} < \sin x$.

2. Given the function $f(x) = \dfrac{4\cos^2 x - 3}{2\cos x}$

   (a) Find the extreme values of $f(x)$ for the interval $\frac{-\pi}{6} \leqq x \leqq 2\frac{1}{6}\pi$, and determine their character.

   (b) Draw the graph of $f(x)$ for $-\frac{\pi}{6} \leqq x \leqq 2\frac{1}{6}\pi$.

   (c) Prove that the graph of $f(x)$ for $\frac{-\pi}{6} \leqq x \leqq 2\frac{1}{6}\pi$ has an axis of symmetry.

3. (a) Draw in one diagram the graphs of the functions

$$F(x) = 2\sqrt{x} \quad \text{and} \quad G(x) = \tfrac{1}{4}x^2 \quad \text{for} \quad 0 \leqq x \leqq 4 .$$

   (b) The line perpendicular to the x-axis and passing through the point on the x-axis with abscissa p, $(0 < p < 4)$, intersects the graph of $F(x)$ at A and the graph of $G(x)$ at B. Find the value of p for which the line segment AB has maximum length.

   (c) The line perpendicular to the x-axis and passing through the point on the x-axis with abscissa 1, intersects the graph of $F(x)$ in C and the graph of $G(x)$ in D. Find the ratio of the two parts in which CD divides the area bounded by the graphs of $F(x)$ and $G(x)$.

ALGEBRA

1. $t_1, t_2, t_3, \ldots, t_k, \ldots$ is an infinite geometric progression, and $t_k = x.p^{k-1}$.

   (a) If $p = 3/5$ and $x = 10$, find t_{11}, correct to two decimal places.

   (b) If $p = 3/5$, find the possible integral value(s) of x such that t_{11} is the first term of the progression smaller than 0.1.

   (c) If $p = \frac{1}{3}x$, the sum of the progression is a function of x. For what values of x is this sum less than 3?

2. Given the function:

$$f(x) = \frac{|x+1| - 1}{|x|} \cdot \frac{x + |x|}{2} \cdot (x-4) \quad \text{where} \quad x \neq 0$$

(a) Find the roots of the equation $f(x) = -1$

(b) Plot the graph of $f(x)$ for the interval $-5 \leq x \leq 5$

3. (a) Solve for x :

$$\sqrt{8x - 16} < x - \tfrac{1}{2}$$

(b) Solve for x :

$$\log_x (10x - 3) - \log_x 3 \leq 2$$

(c) Solve for x:

$$(\tfrac{1}{4}x^2 - x + 1\tfrac{3}{4})^{x(x-4)} < (\tfrac{1}{4}x - x + 1\tfrac{3}{4})^{2(x-4)}$$

ANALYTIC GEOMETRY

1. Given the parabola $y^2 = 2px$ and point P on the directrix of the parabola. F is the focus of the parabola.

(a) Prove that the polar line of P passes through the focus.

(b) The polar line of P intersects the parabola at A and B. Prove that PF is an altitude of triangle PAB.

(c) Find the locus (equation and character) of the midpoints of the chords AB if P moves along the directrix.

2. Given the orthogonal hyperbola $xy = 1$.

(a) Prove that the normals of the hyperbola have positive slopes.

(b) Through an arbitrary point $P(a, \frac{1}{a})$ on the hyperbola two lines are drawn perpendicular to each other. These lines intersect the hyperbola at P and Q, and at P and R respectively. Prove that the line joining Q and R is parallel to the normal at P to the hyperbola.

3. Given the circle $x^2 + y^2 = r^2$ and a point $A(a,0)$, on the positive x-axis, where $a \neq r$. $P(x_1, y_1)$ is a point on the circle. The line OP intersects the right bisector L of AP at S.

(a) Find the equation of L.

(b) Find the equation of the locus of the point S when P moves along the circle.

(c) Specify the locus if :

(i) $a < r$

(ii) $a > r$

SOLID GEOMETRY

1. $ABCD$ is a tetrahedron. $AB = AC = p$, $AD = BC = p\sqrt{2}$. $\angle DAB$ and $\angle DAC$ are equal and acute.

α = angle between the line AD and the plane ABC.

E is the perpendicular projection of B on AD.

(a) Prove that BC is perpendicular to AD, and express the area of $\triangle$ BCE in p and α.

(b) Find α if the tetrahedron has a circumscribed right circular cone with vertex at D.

(c) If α varies, find the locus of the centroid of $\triangle$ BCE.

2. ABCD is a tetrahedron with ABC as its base. $\triangle$ ABC is equilateral. AD is perpendicular to plane ABC. P and Q are the midpoints of AB and AC respectively. The sphere through A,P,Q, and D intersects BD and CD at S and R respectively.

(a) Prove that PQ and RS are parallel, and that the lines AD, PS and QR are concurrent.

(b) Prove that CS is perpendicular to BD.

(c) Prove that the circumscribed circles of $\triangle$ BPS and of $\triangle$ CQR lie on a sphere, and show in a stereometric diagram the location of the centre of the sphere.

3. ABCD EFGH is a cube, with ABCD as its base and EFGH as its top. Edge = p. A right circular cylinder is inscribed inside this cube. The base and the top of the cylinder are the inscribed circles of the squares ABFE and DCGH respectively. K is the midpoint of BF, and N is the point of intersection of DG and CH.

(a) Express in p the distance from the line HK to the axis of the cylinder.

(b) HK intersects the cylinder in K and X. Find the ratio HX : XK.

(c) Consider the squares EKNP when P moves along the inscribed circle of ABFE. Find the location of P for which the volume of the tetrahedron EKNP has maximum value, and express this maximum volume in p.

THE OECTA CUISENAIRE CONFERENCE IN KITCHENER

It was with interest that I attended the debate on the value of the Cuisenaire Method of teaching arithmetic. The Kitchener separate schools have been experimenting with this method for the past two years, and the conference was a discussion of the method.

The Cuisenaire method consists of using ten differently coloured rods of varying lengths, each rod representing from one to ten. It was invented by a Belgian, Georges Cuisenaire, and has been popularized by Dr. Caleb Gattegno, who has written a number of texts on how to teach arithmetic by this method.

On March 1, Dr. Gattegno gave an address about the method. He stated that "For the first time in history, we are giving children the same kinds of problems as research mathematicians. They can now approach mathematics with a freshness of mind." However, he added a word of warning, saying "Anyone who keeps the children too long with the rods is crippling them. The rods are only a model to help chart the way. If they go beyond the generation of ideas, they can be crippling."

The panel debate proved very stimulating. Dr. Floyd Robinson, Research Director of the Canadian Teachers' Federation, termed the Cuisenaire claims exaggerated. He could not agree that it was the greatest scientific step forward since Copernicus, and pointed out that he had already lived through half a dozen world-shaking advances in educational theory. He stated that "The impact of Cuisenaire is as important to children as drugs are on human bodies. If I claim a cure for cancer and sell it, you [Dr. Gattegno] are suggesting that the evidence for it should be supplied by the people who buy it. Poppycock! The onus of the evidence is on those who sell it."

Dr. Gattegno replied: "Why should we measure things that don't measure? When the sun is shining, one doesn't ask for an investigation to prove it is shining. Teachers all over the world are proving to themselves that they get better results in time and with children."

Dr. Robinson stressed he was not opposed to Cuisenaire, only to its unproved claims. He said that many Canadian classrooms are using the Cuisenaire method and that lack of research on it should be viewed with alarm. "I'm very much disappointed that there hasn't been any suggestion to try and set up a trial in a way that will give evidence."

My own reactions to the presentation were mixed; there seems to be no doubt that the use of coloured rods as a teaching aid in the early grades is very helpful, although the price of the rods appears high. I would think that Dr. Gattegno's warning not to prolong the use of the rods unduly is very wise. On the other hand, I felt that Dr. Robinson had an excellent point in urging controlled experimentation to see whether the Cuisenaire method does as much as it claimed to do. Also the thalidomide experience has shown that one should always be on the lookout for possible

side effects in using any product.

Anyone interested in reading about the actual method used in Cuisenaire teaching would be well advised to attend a demonstration or read one of the books by Dr. Gattegno. Perhaps one of the greatest virtues of the method lies in the encouragement for students to learn more in elementary school; this may help combat the "progressive" tendencies of the past thirty years which have tended to oversimplify the curriculum in elementary mathematics.

[Editor's note: Since there is a great deal of material available urging the virtues of the Cuisenaire method, I thought it would be of interest to obtain a more sceptical evaluation. Dr. Robinson has kindly supplied us with a copy of his comments.]

COMMENTS ON THE CUISENAIRE METHOD

by

Dr. F. G. Robinson - Research Director
Canadian Teachers' Federation

OECTA Cuisenaire Conference, Kitchener
March 1-2, 1963

I am well aware that taking the negative side in a Cuisenaire debate is not likely to increase my popularity; in fact, it should be remarked that the committee which set up the panel had considerable trouble locating two people who were foolhardy enough to accept this assignment.

Let me make it quite clear at the outset that I am not a crank who is categorically opposed to the Cuisenaire method or who gets some particular satisfaction out of playing devil's advocate. While not opposed to Cuisenaire, I do feel that the time has arrived when we should examine the rather extreme claims of the proponents of the method in the light of the evidence gathered to date.

The claims I have in mind include, for example, a statement contained in a Cuisenaire advertisement which states that the child will learn more in seven years with Cuisenaire material than he would in twelve years under prevailing methods[1], and the even more startling promises which one finds in press and magazine reports. Many of my colleagues in the field of educational research are concerned that the rate of promotion and expansion of the Cuisenaire method have vastly outstripped the rate of gathering acceptable evidence regarding its effectiveness.

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[1] Dr. Gattegno has recently reduced his estimate of the time required to six years.

You may well ask what I mean by "acceptable evidence". I would answer that while complete "proof" will probably always be beyond our grasp in education, we can nevertheless make comparisons between how much children learn through the Cuisenaire technique and how much they learn by methods presently in vogue. But before such comparisons can be regarded as "acceptable evidence", they must take account of a number of important factors which are undoubtedly related to the amount which children will learn under any technique.

For example, in making such a comparison we must be sure to allow equal time to both methods; this may seem an obvious point, but Brownell has reported that many British schools who are getting extraordinary results with the Cuisenaire method are devoting a good deal more time to arithmetic than would normally be allotted (and I know this has happened in at least one large Canadian centre as well). Moreover, reliable comparisons must take into account possible differences in student ability in the classes being compared; ideally, these differences should be removed (or taken account of) by a proper experimental design. Most important of all, of course, valid comparisons must take account of the fact that teachers themselves, no matter what technique they are using, vary tremendously both with regard to enthusiasm and efficiency.

When we come to examine the research literature we find that remarkably few pieces of "acceptable evidence" have been made public. I think the state of affairs should be looked upon with some alarm, since the Cuisenaire method has apparently been in use for approximately ten years and current reports indicate it is now installed in at least hundreds (and possibly thousands) of Canadian classrooms. The results of the few studies which have been published, and which have given any attention whatsoever to the factors I touched upon earlier, can be summarized as follows:

1. A number of reports of tryouts at the Grade 1 level (and there are probably no more than eight of these reports at the outside) have indicated that Cuisenaire students markedly outperformed students taught by conventional methods in complex arithmetical computations involving fractions, but that the two groups performed about equally well on the basic number work which has normally been taught in Grade 1, and on tests which require the student to apply his knowledge to solve problems. While the comparisons supporting these conclusions were by no means perfect, I think that the findings have to be given some weight.

2. A very few studies which have tried to trace Cuisenaire-trained children through Grades 2, 3, or 4, have purported to show that these students made a continued gain in the more complex computations, and have even on occasion claimed some advantage for Cuisenaire students in the basic number work. However, these studies are so sketchily reported, and this kind of comparison is so fraught with difficulties, that I would not be prepared to say just what one should read into these reports.

Three observations are in order concerning these results. In the first place, the quantitative measurements have yielded results which, while apparently favourable to the Cuisenaire method, by

no means match the extravagant claims which I cited earlier. In fact, a cynic might observe that the more rigorously conducted the experiment, the more modest the resulting claim has been. Moreover, the conclusions in these reports must be regarded as very tentative because they are based on comparisons which by no means come to grips with all the relevant variables; for example, one will find a report which compares Miss Smith's Cuisenaire class with Miss Jones' conventional class, while remarking, rather innocently, that the teachers were "selected" or "matched" to be of equal ability. Another very relevant point is that the one area in which an indisputable gain has been made seems to involve complicated arithmetical computations which no sensible mature person would likely undertake and which in many ways seem to have been contrived to fit the kind of learning which follows most directly from the method; meanwhile, little or no gain has been demonstrated in the basic number work of arithmetic nor in problem solving.

Even if we accept for the moment that the Cuisenaire classes seem to be out-performing normal classes, we still have to ask whether this better performance is due to some intrinsic superiority of the method or to some other factor which cannot be dealt with in the kinds of comparisons which we are making. For example, we all recognize that our school systems, and the arithmetic programs for these school systems, badly underestimate what the child can do. In the conventional classroom we will often find a not-too-enthusiastic arithmetic teacher who defines her class goals in terms of the presently prescribed grade standard. On the other hand, in Cuisenaire studies we have extremely enthusiastic teachers who have been given considerable freedom to experiment, and more important, to set their own level of accomplishment for their classes. As Brownell has sagely remarked, it may well be that it is the higher goals set by the enthusiastic Cuisenaire teachers rather than the intrinsic merit of the method itself which accounts for the additional learning. He goes on to add that teachers using conventional techniques could well set higher goals and achieve them too.

In short, until better evidence is forthcoming (preferably something based upon recognized experimental procedure), I am not prepared to accept the claim made by Cuisenaire enthusiasts that the method represents a fundamental breakthrough in education which will ensure spectacular results. I have been much too close to another widely acclaimed "breakthrough" called programmed instruction (whose advocates heralded it as "the most significant development in education since the invention of the printing press") not to have learned that the extravagant claims of the early years of a new idea must usually be tempered when acceptable evidence starts to accumulate. Frankly, I think that the learning process is still largely a mystery and will likely remain a mystery and that rather than expecting fundamental breakthroughs we should reconcile ourselves to a series of smaller, yet important, advances; it is my belief that programmed instruction and the Cuisenaire method will both find a significant place in this series of small advances.

It also seems that Cuisenaire proponents should attempt to formulate a theory as to why the method works, a theory which can be reconciled with, or at least related to, conventional psychological theory. I have studied all of the Cuisenaire publications (which, incidentally, are overlapping and phrased in a vague quasi-psychological language), and I find some brief, indirect references to the work of Piaget and criticisms of teaching methods based on counting procedures, I could not find a coherent set of acceptable psychological principles which would explain in what way the Cuisenaire method is uniquely related to the fundamental psychology or thinking processes of the child. I am particularly at a loss to explain the significance of colour in number concepts except to say that it may act as an additional cue or crutch. This is a significant issue because many people of a mathematical bent have criticized the Cuisenaire method's dependence on colour, which they regard as an extraneous non-mathematical factor. Certainly I can find no experimental evidence whatsoever that the colouring of the rods is crucial to the success of the method.

In short, despite an honest effort, I really must confess that I do not understand the "theory" of the Cuisenaire method. While it is of no consequence that I do not understand it, it is of considerable importance and concern that Cuisenaire teachers do not appear to understand it either. Unless these teachers can make themselves understood to honest inquirers with some command of psychological and mathematical concepts, they leave themselves open to the charge that they belong to a "cult" rather than to a scientific discipline.

Conclusion

There are many other questions which could, and probably should, be asked if time permitted. For example, what evidence is there that the Cuisenaire system is superior to the Stern, Montessori, Shaw, Unifix, Avon, Bass, Arnold, or any number of other systems which might be named.[2] Let me answer this question by saying that there is no such evidence, and the fact that the Cuisenaire method has practically a clear field in Canada should be taken as a tribute to Dr. Gattegno's promotional ability, but not necessarily an indication of any intrinsic superiority of the method. It should also be noted that the important long-range effects of Cuisenaire training are not known. Whether or not Cuisenaire-trained children will become relatively more competent mathematicians, for example, is an open question.

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[2]It should be noted that a survey conducted in 1959 (i.e., approximately six years after the introduction of the Cuisenaire method) indicated that the Cuisenaire method was being employed in only 3 per cent of English schools, while other non-traditional forms of arithmetical instruction were employed in 22 per cent.

In summary let me reiterate that I am by no means opposed to the Cuisenaire method; in fact, I would advocate that schools experiment with it. I use the term "experiment" advisedly, because I regard the education of my own children to be of such importance that it must be considered inexcusable for a teacher or a school to rush willy-nilly into this new venture with no other concrete evidence regarding its effectiveness or consequence than their raw enthusiasm. I would strongly urge that any teacher who plans to try the Cuisenaire method should make an effort to arrange the tryout so that reliable evidence will be forthcoming.

[Editor's Note: We have been fortunate enough to obtain comments from the three Examiners-in-Chief of the Grade XIII Mathematics examinations; these will appear in this and succeeding issues.]

GRADE XIII GEOMETRY - 1963

(by the Examiner in Geometry)

Slightly less than one quarter of the 1963 paper consisted of book-work, compared with 36% on the 1962 paper. However, the other questions on the 1963 paper were easier than in 1962 and the results were somewhat better. In all 15,400 papers were marked, an increase of 1000 over 1962. The marks ranged all the way from zero to one hundred; about half a dozen students made one hundred per cent.

To be successful at analytic geometry a student requires a fair knowledge of both elementary algebra and synthetic geometry. It is a constant source of wonder that students should have such a weak memory for the algebra taken up in grades X and XII. This was noticed throughout the paper but particularly in the attempts to solve the quadratic equation in question 5, and in the inability to recognize in question 3, that $x^2 - xy = 0$ implies that $x = y$ or $x = 0$. The knowledge of synthetic geometry appeared to be greater, but many students seemed unable to recall that a median bisects the area of a triangle, a fact that was useful in question 2(d).

The first four questions on the paper were fairly easy and of gradually increasing difficulty. These questions were generally handled rather well. Questions 5 and 6 were of intermediate difficulty while 7, requiring a proof that a hyperbola has two asymptotes (together with an easy problem on asymptotes) and 8, a problem on conjugate diameters, made up the difficult part of the paper.

Most students answered 7(a) by showing that the lines $bx \pm ay = 0$ possess the property that the perpendicular distance from any point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the appropriate line, tends to zero, as the abscissa of the point tends to infinity. According to a strict interpretation of the wording of the question, it would have been better to use the other definition of an asymptote to a curve, and show that a tangent at any point on the hyperbola tends to one or other of the lines $bx \pm ay = 0$, as the point of contact tends to

infinity. It is unfortunate that tangents to hyperbolas are not in the Grade XIII course of study. The tangent approach was used by a number of students, but in most cases was invalidated by incorrect procedures for taking limits. It might possibly help many students (and teachers) if the formal definition of limit was discussed in connection with this theorem. This could possibly remove some of the mystical atmosphere surrounding the notion of limit and, in addition, serve to illuminate the theorems dealing with tangents to conics. The invalid proof of this theorem given in the unrevised edition of Durrant and Kingston's book was still used by a considerable number of candidates.

The final question on the paper (traditionally considered as the scholarship question) was not too difficult, and was attempted by about 20% of the students, although only 2% produced a complete solution.

THE O.S.S.T.F. DISTRICT TEN MATHEMATICS CONTEST

William Nediger (Clinton)

As you are undoubtedly aware, each year the Mathematical Association of America, in conjunction with the Institute of Actuaries, sponsors a contest for senior high school students of mathematics. A great many schools in Ontario are already having many of their Grade XII and Grade XIII students write this contest.

Last February at the District 10 Educational Conference, a group of four teachers, Bill Nediger (Clinton), Ron Dunkley (St.Mary's), Don Attridge (Orangeville), and Ed Anderson (Wingham), discussed the possibility of a similar contest at the junior level for students from Grades IX, X, and XI. From this discussion a contest for District 10 schools evolved.

The contest format was modelled on that of the senior contest. The problems were set by the members of the committee and the marking was handled at Seaforth under the direction of Bill Nediger. Participation in the contest was almost 100%. One school held a preliminary contest with 120 students to decide which 30 might write the actual contest. The four highest schools in the district (the team score being the total of the top 3 in each school) were: Wingham, St. Mary's, Goderich, Dublin.

The three top students from each school were rewarded by attendance at a seminar held at the University of Waterloo. These students were also to be suitably recognized by their respective schools. Such gifts as books or slide rules were given. The University also donated a permanent shield, which passes for one year to the custody of the winning school, and book prizes for the top three individuals in the entire contest.

Various other districts have inquired about the possibility of using the District 10 Contest in their districts. The District 10 Committee would like to take this opportunity to assure any other districts that District 10 would be glad to make this con-

test available each year to any district which would like to use it. The Committee feels that, in order for a district to be eligible, 60% of the schools in that district should enter. Representatives from each of the 31 districts have been contacted. These representatives have agreed to organize the contest in their district. They will distribute the contest and collect the answers which they will then mail to a central location in District 10 for marking.

Results will then be forwarded to each participating school.

The Committee feels that this should be a contest within each district. The University of Waterloo has agreed to donate a shield for each district, as has been done in District 10. A shield for the provincial winner has also been made available by the University. An invitation is extended to all districts to arrange for a seminar at Waterloo. Where this is impossible because of distance, a speaker would be provided by Waterloo for a local seminar at the ceremony of presentation of the shield. Of course, a district would be quite free to organize a seminar with other local arrangements.

Copies of last year's contest are available at a price of $1.00 for 10 copies (copies are sold only in multiples of 10). These should be ordered through your district representative no later than Dec. 22, 1963. The contest will be held on the first Wednesday after the Easter vacation.

The following is the list of Chairmen for the districts. Where no name is listed, teachers have been asked but no reply has yet been received. Anyone willing to act in this capacity for such a district should contact Ron Dunkley at St. Mary's Collegiate, St. Mary's, Ontario.

Chairmen who have so far agreed to act.

District 1.	Keith Dawson - Wallaceburg D.S.S.
2.	Charles Robinson - Central Elgin C.I.
3.	
4.	Wilfred Baxter - Galt-Glenview Park S.S.
5.	
6.	Jack Vollmer - Paris District H.S.
7.	Lorne Smith - Laurence Park
8.	Robert McNeil - Beamsville D.S.S.
9.	Archie Foss - N. Falls Coll.Voc.Inst.
10.	Claude Kalbfleisch - Goderich
11.	Jim Russell - Collingwood Coll. Inst.
12.	Ken Jackson - Georgetown
13.	William Gartrell - Fort William C.I.
14.	Henry Dearborn - Burnhamthorpe C.I.
15.	
16.	
17.	Neil MacEwan - Patterson C.I.
18.	
19.	James Fencott - R.H.King C.I.
20.	
21.	

22.
23.
24. Helen Hay - Queen Elizabeth C. and C.I.
25. Jack Arnatt - Cobden D.H.S.
26. Don Fox - Bell High School
27.
28.
29.
30. David Katz - Nelson
31. David Mavin - Stouffville

Correspondence on Asymptotes

G. Berman

In the penultimate issue of the Gazette, the article on "Asymptotes" evoked some correspondence. Three communications of general interest should be summarized.

1. Russell Phillips (U.T.S.) points out that the method illustrated in the article for finding asymptotes of the form $y = mx + b$ does not give vertical asymptotes. These must be determined by inspection. For example, the curve

$$y = \frac{3x^3 + 7x + 1}{(x - 1)(x + 5)}$$

has $x = 1$ and $x = -5$ as vertical asymptotes.

2. W. F. Elliott (Stayner) asks for an illustration of a curve in which the curve constantly lies on the same side of the pseudo-asymptote. This is provided by the curve

$$y = \frac{1}{x} + \left|\frac{\sin (x^2)}{1000x}\right|$$

where $x > 0$. As x becomes infinite, the curve approaches the x axis in position, always remaining above the x axis. The slope of the tangent does not, however, approach a limit.

3. O. Zirke (Royal York C.I.) points out an error in our first example concerning the curve $y = \frac{\sin x}{x}$. The equation of the tangent at the point with abscissa a is readily found to be

$$y - \frac{\sin a}{a} = \left(\frac{\cos a}{a} - \frac{\sin a}{a^2}\right)(x - a)$$

and the y intercept of this tangent is obviously the point $(0, -\cos a + 2\frac{\sin a}{a})$. No matter how far the point $(a, \frac{\sin a}{a})$ recedes from the origin, the tangent will not approach a limiting position; its point of intersection with the y axis will oscillate back and forth between $+1$ and -1. Consequently, the x axis is not an asymptote to the curve (in our original example, we did not go far enough; we checked only on the position and slope properties of the tangent).

4. As a development of Mr. Zirke's discussion, it might be worthwhile to indicate the general result of which it is an illustration. One may prove the

Theorem: The line $y = Ax + B$ is an asymptote of the curve $y = f(x)$ if and only if

$$\lim_{a\to\infty} f'(a) = A, \quad \lim_{a\to\infty} [f(a) - a\, f'(a)] = B.$$

Proof: The tangent at $[a, f(a)]$ is

$$y - f(a) = f'(a)(x - a).$$

This may be written as

$$y = f'(a)x + [f(a) - a\, f'(a)].$$

This line approaches a fixed limiting position $y = Ax + B$ if and only if the y intercept

$$B = \lim_{a\to\infty} [f(a) - a\, f'(a)]$$

and the slope

$$A = \lim_{a\to\infty} f'(a).$$

If we consider the case $A = B = 0$, we get the

Corollary 1. The line $y = 0$ is an asymptote of the curve $y = f(x)$ if and only if

$$\lim_{a\to\infty} f'(a) = 0, \quad \lim_{a\to\infty} [f(a) - a\, f'(a)] = 0.$$

Corollary 2. The line $y = 0$ is an asymptote of the curve $y = f(x)$ if and only if

(1) $\lim_{a\to\infty} f(a) = 0$

(2) $\lim_{a\to\infty} f'(a) = 0$

(3) $\lim_{a\to\infty} a\, f'(a) = 0.$

[Note that in our earlier "Example 1", we checked only the first two of these required conditions.]

Problem 7. Given $\triangle ABC$ and median AD. P is <u>any</u> point on AD. BP is produced to meet AC at F and CP is produced to meet AB at E.

Prove $EF \| BC$.

BOOK REVIEWS

MODERN ELEMENTARY ALGEBRA - Nichols and Collins
(Holt, Rinehart, Winston)

The text opens with an introduction to sets, with emphasis on matching of sets, Venn diagrams, union and intersection of sets. Problems with sets continue to advantage throughout the text.

Operations with numbers follow with the basic laws applied to positive rational numbers, integers, rationals, and irrationals. The introduction to irrationals at this early stage, to complete the set of real numbers, is incidental in treatment, and they are rarely mentioned again until the closing chapters of the text. Even at this early stage, a working knowledge of the Theorem of Pythagoras is presumed.

The use of such symbols as squares, circles, triangles, first as place holders and later as variables, persists throughout most of the text; while their purpose is not questioned, they have no intelligible oral equivalent and appear to be overworked.

The programme of the text progresses logically and smoothly. Explanations are thorough and the exercises are exceptionally good. Inequalities and range of the variable are dealt with more adequately than in many contemporary texts. The topics covered are approximately the equivalent of the Algebra through grades nine to twelve of the present Ontario curriculum.

This is essentially a student's text and, as such, is outstanding in format, language, presentation, and exercises. The use of two-colour print is not excessive; it is used for chapter and topic headings. A cumulative review of questions is included at the end of each chapter; these are more of a remedial than of an enrichment nature. A bibliography is included at the end of each chapter. No answers to problems are provided, but I do not mention this fact as a criticism. The text concludes with a sound introduction to polynomials, functions, and relations.

Two criticisms of the contents are the use of m and n as variables in the graphing of straight lines, and the confusing and inept treatment of the slope of a line in Chapter Ten.

As a source book for problems and as a reference book for clear-cut presentation of elementary Algebra, I would recommend this text highly.

James D. Smith, Selkirk C.I. and V.H.S.

INTRODUCTION TO GEOMETRY - H.S.M.Coxeter (John Wiley and Sons,N.Y.)
(obtainable at the University of Toronto Bookstore)

This outstanding book on geometry should be in the library of every secondary school in Ontario; in fact, larger schools which have enriched classes may want several copies for reference.

The book is divided into four parts. Part 1 discusses triangles

polygons, circles and spheres, similarity, crystallography, and other topics, by synthetic methods. Particularly outstanding are the discussions of the nine-point circle, inversion, Sylvester's problem on collinear points, Morley's theorem that the trisectors of the angles of any triangle intersect to give an equilateral triangle.

Part 2 introduces Cartesian co-ordinates and has excellent chapters on complex numbers, the five Platonic solids, and the golden (medial) section. The work on Euler's formula, the Fibonacci numbers, and phyllotaxis is especially interesting.

Parts 3 and 4 are more difficult; part 3 discussed various types of geometry (ordered, affine, projective, absolute, hyperbolic) and Part 4 discusses tensor notation and differential geometry.

Parts 1 and 2 contain the material which will be of most interest to teachers and superior students. In these parts, the book is a gold mine of information, and the manner of presentation is very vivid and stimulating.

N.N.H.

NUMERICAL METHODS FOR SCIENCE AND ENGINEERING - R.G.Stanton
(Prentice-Hall, Inc., Englewood Cliffs)
(obtainable at the University of Waterloo Bookstore)

This is another Ontario book which I feel would be of value in all secondary school libraries for reference and enrichment.

The first chapter introduces finite difference notation and gives plenty of exercise in the use of sigma notation and mathematical induction. Students will find the fact that polynomials of degree n give constant n'th differences a simple but interesting development.

Other topics in the book include interpolation, numerical computation with integrals and series, numerical solution of systems of linear equations, numerical solution of differential equations. Sections of particular use at the senior high school level might be: finding best-fitting straight lines by the method of least squares, graphic solution of equations of the form $f(x) = 0$, numerical methods for solving such equations, some of the work on series, the Simpson method for finding an area, the excellent survey of determinants and their application to linear equations. The "relaxation method" for solving linear equations is discussed very lucidly, and could certainly be used as a supplementary topic even in early grades.

N.N.H.

THE RADICAL AXIS

E. Orlando (Ottawa)

If one refers to either of the two texts in common usage in Ontario, one finds that the topic of radical axis is loosely handled; the most universal conclusions that would tie things together nicely are sadly lacking.

Viz: (1) The general proof of the theorem: "The radical axes of any three circles are concurrent."

(2) The accurate construction of the radical axis of 2 non-intersecting circles.

(3) The fact that, for 2 non-intersecting circles, $C_1 + kC_2 = 0$ represents a family of circles of which all possible pairs have the same radical axis (the texts deal only with the case of intersecting circles).

Better students have always asked questions about the work in this area that could only be answered by solving the above 3 problems. Consequently, in my own classes, I have, in recent years, added these theorems to the course of study.

(1) Given: 3 circles represented by $C_1 = 0$, $C_2 = 0$, and $C_3 = 0$. Required to prove that the radical axes are concurrent.

Method I - ANALYTIC

Solution: Define $P_1(x_1,y_1)$ as the point of intersection of the radical axes represented by $C_1-C_2=0$ and $C_2-C_3=0$.

Since P_1 lies on both of these lines, the expression C_1-C_2 vanishes for the coordinates of P_1. C_2-C_3 also vanishes for the coordinates of P_1. Hence $(C_1-C_2)+(C_2-C_3)$ also vanishes for the coordinates of P_1, that is, $C_1-C_3 = 0$ is satisfied by the coordinates of P_1. We conclude that the radical axis represented by $C_1-C_3 = 0$ also passes through P_1. Thus, we have shown the three radical axes are concurrent.

Method II - SYNTHETIC

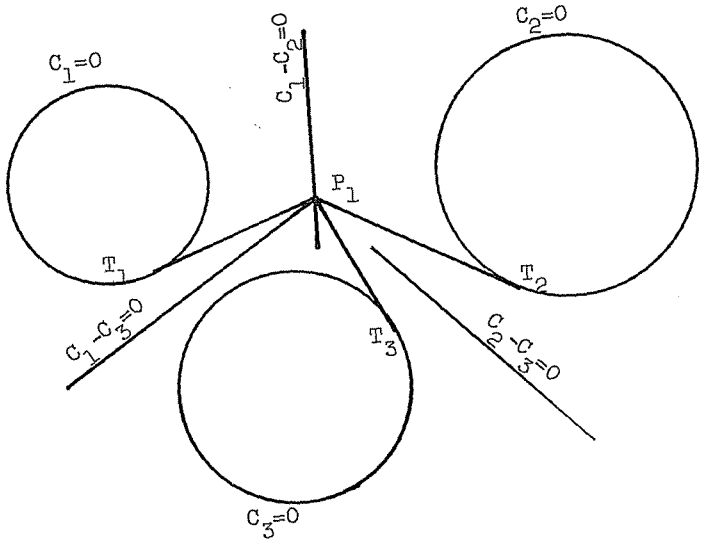
(P_1 is defined as in Method I.)

Since P_1 lies on the radical axis by $C_1-C_2 = 0$, $P_1T_1 = P_1T_2$

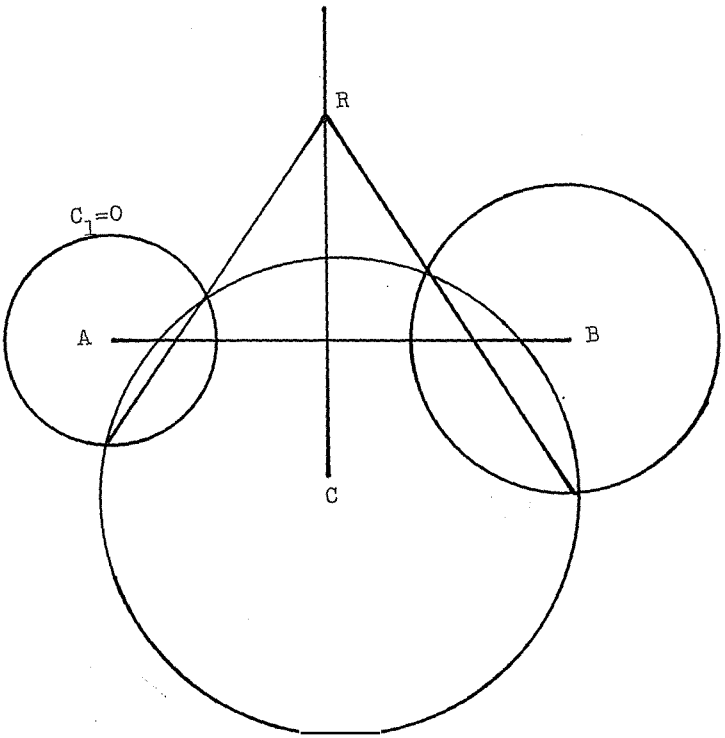
Also P_1 lies on the radical axis represented by $C_1-C_3 = 0$;

hence $P_1T_1 = P_1T_3$. Thus $P_1T_2 = P_1T_3$.

It follows that P_1 lies on the radical axis of the second and third circles; thus the three radical axes are concurrent.



(2) We may now use the preceding theorem to construct the radical axis of two non-intersecting circles.



We are required to construct the radical axis of 2 given circles with centres A and B. We first draw a 3rd circle which intersects each of the given circles. Since, for intersecting circles, the radical axis is the common chord produced, the radical centre R is the point of intersection of the lines of the two common chords. It is also fundamental that the radical axis of 2 circles is perpendicular to the line of their centres. Hence, a line through R, perpendicular to AB is the radical axis of the two given circles.

(3) A more general interpretation of the locus represented by $C_1 + kC_2 = 0$.

[Note: C represents the expression $x^2 + y^2 + 2gx + 2fy + c$]

The equation $C_1 + kC_2 = 0$ (k a parameter, $k \neq -1$) represents a family of circles through the intersection of the circles represented by $C_1 = 0$ and $C_2 = 0$. This is not the most universal truth about such an equation since it pertains only to intersecting circles. It is only natural to ask what it represents in the case of non-intersecting circles. Before attempting to answer this let us consider a few basic fundamentals.

When $k = -1$, $C_1 + kC_2 = 0$ reduces to $C_1 - C_2 = 0$ which is the equation of the line of the common chord of the two given circles. $C_1 - C_2 = 0$ also represents the equation of the radical axis of any pair of circles whether they intersect or not. Hence, in the case of two intersecting circles, the common chord produced turns out to be also the radical axis of the circles. Intuition would suggest that, for any pair of circles, $C_1 + kC_2 = 0$ represents a family of circles of which every possible pair has the same radical axis, namely, the line represented by $C_1 - C_2 = 0$.

The equation of the radical axis of this pair is :

$$(1 + k_2)(C_1 + k_1C_2) - (1 + k_1)(C_1 + k_2C_2) = 0,$$

$$C_1(1 + k_2 - 1 - k_1) + C_2(k_1 + k_1k_2 - k_2 - k_1k_2) = 0$$

$$C_1(k_2 - k_1) - C_2(k_2 - k_1) = 0$$

$$C_1 - C_2 = 0$$

Hence, we see that our hypothesis is valid.

Prof. R. G. Stanton, Editor, Ontario Mathematics Gazette,
University of Waterloo, Waterloo, Ontario.

Dear Editor,

In Volume 2 number 1 issue of the Ontario Mathematics Gazette there appears to be a difference of interpretation of problem 4 of the Problems Paper solutions for 1962. The following solution should illustrate the other interpretation.

The dividend of \$15000 should include both the interest at 9% on the investment \$P and a sum \$R to be invested at 4% for replacement of the mine in 25 years.

$\therefore \quad 15000 = .09P + R$

Now $R\,(1 + 1.04 + 1.04^2 + \ldots + 1.04^{24}) = P$

$\therefore \quad P = R\,\dfrac{(1.04^{25} - 1)}{.04}$ or $R = \dfrac{.04P}{1.04^{25} - 1}$

$\therefore \quad 15000 = .09P + \dfrac{.04P}{1.04^{25} - 1}$

Solving for P we get the value of the mine to be \$131,570.

A third solution to question 2 might also be of interest. Since the sequences contain 0,1,2,3,...,r, B's then the number of sequences is $N = {}_nC_0 + {}_{n+1}C_1 + {}_{n+2}C_2 + \ldots + {}_{n+r}C_r$.

$${}_nC_0 + {}_{n+1}C_1 = {}_{n+2}C_1$$

$${}_{n+2}C_1 + {}_{n+2}C_2 = {}_{n+3}C_2$$

$${}_{n+3}C_2 + {}_{n+3}C_3 = {}_{n+4}C_3 \quad \text{and finally} \quad {}_{n+r}C_{r-1} + {}_{n+r}C_r = {}_{n+r+1}C_r$$

I hope that the Gazette may continue to serve the mathematics teachers in the province.

Yours sincerely,
J. Norman C. Sharp,
North Albion Collegiate

BAD FORM IN MATHEMATICS

by a former Examiner-in-Chief

Every year, hundreds of marks are lost on Grade 13 examinations on account of BAD FORM. It is the purpose of this set of examples to point out some of the more common pit-falls into which unwary and sloppy students are constantly falling.

1. MISUSE OF BRACKETS

This is the commonest type of bad form. The trouble is that students omit brackets, then forget they have omitted them, and carry on incorrectly.

Example 1. Given $f(x) = x^2 + (3-2p)x + 17p-1$, compute $f(-2)$.

$$f(-2) = (-2)^2 + (3-2p)-2 + 17p-1 \qquad \text{BAD FORM}$$
$$= 4 + 3 - 2p - 2 + 17p-1 \qquad \text{LEADS TO}$$
$$= 4 + 15p \qquad \text{ERROR}$$

Example 2. Students frequently write $f-2$ in Example 1, when they mean $f(-2)$.

Example 3. Students frequently write $\dfrac{\frac{2}{3} - \frac{1}{3} - \frac{4}{3}}{3!}$

when they mean $\dfrac{(\frac{2}{3})(\frac{-1}{3})(\frac{-4}{3})}{3!}$

Example 4. Students frequently write $x-a \cdot f(x)$ when they mean $(x-a)\, f(x)$ or $(x-a).f(x)$.

Example 5. $p^3+q^3 = (p+q)(p^2+q^2-pq)$
$= (p+q)(p+q)^2 - 3pq$
$- (p+q)^3 - 3pq$

Students must learn to use several kinds of brackets, and write:
$p^3 + q^3 = (p+q)[(p+q)^2 - 3pq]$.

Example 6. Given $f(x) = 2x - 3 + 5x^2$, compute $f(\frac{1}{2})$.
"Solution": $f(\frac{1}{2}) = 2(\frac{1}{2}) - 3 + 5\frac{1}{4}$
$= 1 - 3 + 5\frac{1}{4}$
$= 3\frac{1}{4} = \frac{13}{4}$.

Note how the sloppy omission of the parenthesis about $\frac{1}{4}$ led to an error.

Example 7. $x = \frac{4}{3}y$
$y = \pm 2$
Hence $x = \frac{4}{3} \pm 2 = \frac{4 \pm 6}{3} = \frac{10}{3}$ OR $-2/3$.

Note again that omission of brackets has again led to error; the correct answer is, of course,
$$x = \pm \frac{8}{3}.$$

Example 8. "The line passes through the point 3,-2."

This is bad form; the co-ordinates of the point must be written as $(3,-2)$.

2. PERENNIAL USE OF THE EQUALS SIGN

The sign = does not mean is.

Example 9. "The equation $= 3x^2 - 2x + 5 = 0$."

The equation is not equal to zero; in fact, the whole phrase "$3x^2 - 2x + 5 = 0$" is the equation. The first = sign should be replaced by the word is.

Example 10. Solve $3x-2 = 5 - (9-8x)$.

"Solution": $3x-2 = 5-(9-8x)$

$= 3x-2 = 5-9+8x$

$= 3x-8x = 5-9+2$

$= -5x = -2$

$= 5x = 2$

$= x = \frac{2}{5}$

This solution is worth zero marks. In fact, if you trace through it, you see that the student has $2 = -2 = \frac{2}{5}$ as part of his work.

This sort of nonsense can be avoided by omitting the 5 = signs at the beginnings of the lines.

3. LACK OF LOGIC

Some statements just do not make sense.

Example 11. $y = 3x^2 - 2x + 1$.

$\frac{d}{dx} = 6x - 2$.

This is silly, since $\frac{d}{dx}$ can not stand alone here. Write $\frac{dy}{dx}$ and read "the derivative of y with respect to x".

Example 12. Prove that $\csc^2 x + \sec^2 x = \csc^2 x + \sec^2 x$.

"Solution": $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{1}{\sin^2 x} \frac{1}{\cos^2 x}$

$\frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x}$

$\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x}$

Hence the identity is true.

The "solution" is of little value, since it starts out by stating as a fact what is required to be proved. The "solution" can be corrected by writing the words TRUE IF at the beginning of each of the three lines.

Example 13. $3x - 2y = 0$
$7x + 9y = 2$

"Thus the first equation is zero." This statement is silly. The first equation reads $3x - 2y = 0$, and the only correct statement of this nature is: "The right-hand side of the first equation is zero".

[Continued on page 4]