8 PAGE INSERT ELEMENTARY MATH
Abacus

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 53 #3
March 2015

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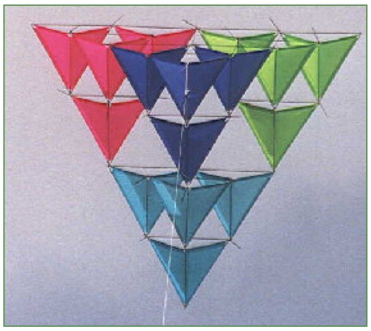

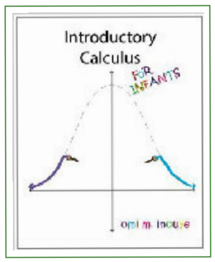

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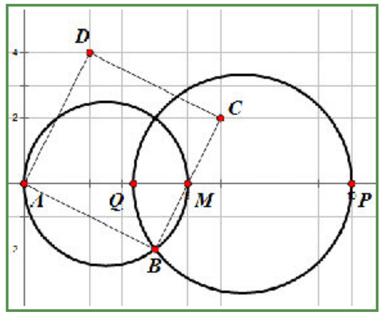

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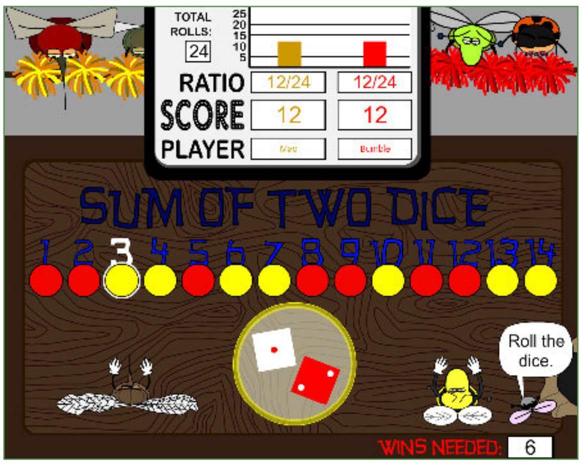

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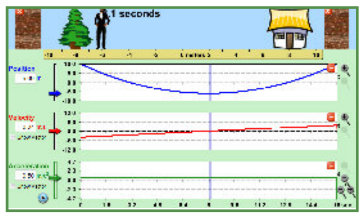

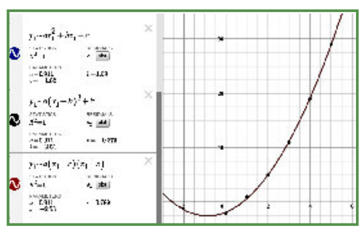

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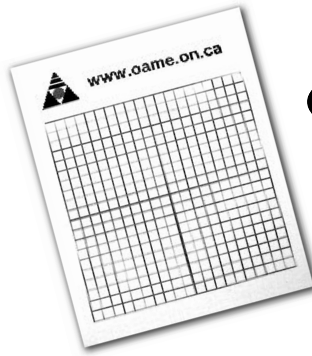

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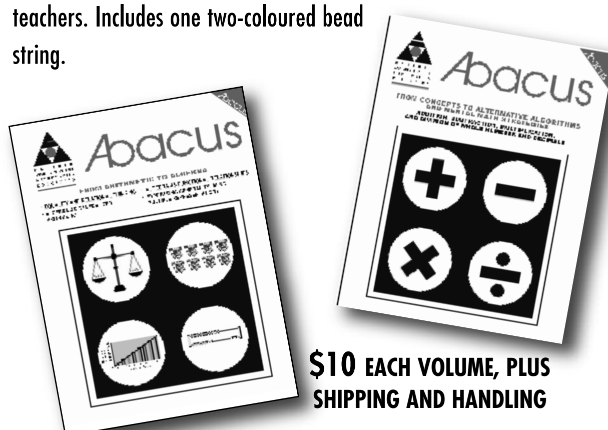

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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Dan Jarvis, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Dan Jarvis at **dan.jarvis@oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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▲ About the *Ontario Mathematics Gazette*

The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

Submission of Advertisements

Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** at the above address. Courier is recommended to avoid possible delays. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

Advertising Rates

Advertising rates are available by telephoning, e-mailing, or writing to the Advertising Manager.

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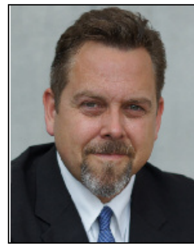
▲ Editor's Message



DANIEL H. JARVIS, O.C.T., Ph.D.
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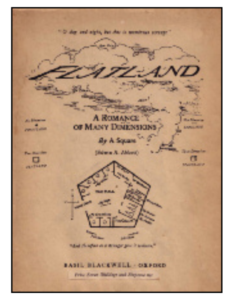
http://faculty.nipissingu.ca/danj/

Dr. Daniel Jarvis is Professor of Graduate and Mathematics Education in the Schulich School of Education at Nipissing University, North Bay, Ontario. His research interests include instructional technology, integrated curricula, and mathematics of the workplace.

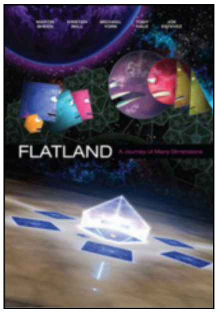
Greetings once again, fellow math teachers, coordinators, researchers, and general math enthusiasts!

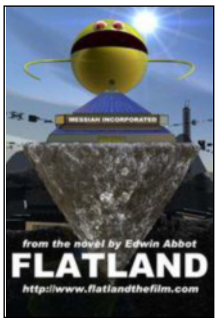
In 1884, English schoolmaster, Edwin Abbott, in his intriguing satirical novella entitled *Flatland: A Romance of Many Dimensions*, presented the life and adventures of the main character, A. Square, attorney-at-law, who inhabits a two-dimensional world, yet is one day visited by a three-dimensional guest, A. Sphere. The book used the fictional two-dimensional world of Flatland to comment on the hierarchy of Victorian culture, but also provided a careful and delightful examination of multiple dimensions. What follows is the classic meeting of the above-mentioned protagonists:



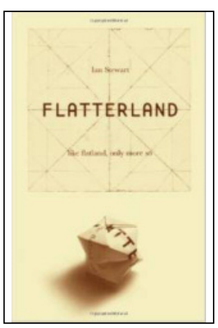
> **Stranger:** Now, Sir; listen to me. You are living on a Plane… I am not a plane Figure, but a Solid. You call me a Circle; but in reality I am not a Circle, but an infinite number of Circles, of size varying from a Point to a Circle of thirteen inches in diameter, one placed on the top of the other. When I cut through your plane as I am now doing, I make in your plane a section which you, very rightly, call a Circle. For even a Sphere—which is my proper name in my own country—if he manifests himself at all to an inhabitant of Flatland—must needs manifest as a Circle… [Y]our country of Two Dimensions is not spacious enough to represent me, a being of Three, but can only exhibit a slice or section of me, which is what you call a Circle.
>
> **A. Square:** Every reader in Spaceland will easily understand that my mysterious Guest was

speaking the language of truth and even of simplicity. But to me, proficient though I was in Flatland Mathematics, it was by no means a simple matter. (pp. 85–86)





In 2007, Flatland was made into two separate computer-animated films. *Flatland: The Movie*, directed by Johnson and Travis, and produced by Caplan and Wallace, was created as a more kid-friendly version, straying farther from the original text, and side-stepping or down-playing many of Abbott's more controversial social issues. In this film, Flatlanders have fractals for their insides, and carry suitcases with them by some magical means. In contrast, *Flatland: The Film*, directed by Ladd Ehlinger Jr., and produced by F.X. Vitolo, was made more for an adult audience (e.g., some animated violence, as mentioned in the book), and preserved the social dynamics as described in Abbott's world. In this version, we can see Flatlander's internal organs, the clockwork of their brains and hearts, and wiggling hairs which cover their bodies and presumably aid in locomotion within their flat world. Abbott's *Flatland* novella was apparently among Einstein's favourite texts, the main theme of dimension visualization undoubtedly parallelling his own pursuit of attempting to describe a fourth dimension (i.e., time) to the lay reader, when elaborating upon his theories of relativity.



Dr. Ian Stewart, a Professor of Mathematics at the University of Warwick, published a sequel to Abbott's classic work entitled Flatterland: *Like Flatland, Only More So* (2001). The multi-dimensional plot line runs as follows. Nearly a century after A. (Albert) Square's adventures, his great-great-granddaughter, Victoria Line (Vikki), finds a copy of his book in her basement. This prompts her to invite a sphere from Spaceland to visit her, but instead she is visited by the "Space Hopper" who can actually travel to any space in the Mathiverse, a set of all imaginable worlds. After showing Vikki higher dimensions, he begins showing her more modern theories, such as fractional dimensions, dimensions with isolated points, topology, hyperbolic geometry, the Projective "Plain," and the quantum level.

An excerpt from the book:

> The Peoples of Planiturth are just as complacent about living in a 3D world as we Flatlanders are about living in a 2D one. Most of them are convinced that there is no such thing as the Fourth Dimension. The Space Hopper says that they're right, but for the wrong reason. I wonder what he means by that?... "They're right, Vikki, because there is no such thing as the Fourth Dimension. And they're wrong, because there are lots of different Fourth Dimensions—not to mention Fifth, Sixth, or even a Hundred-and-First—many of which they experience in their daily lives, but fail to recognize." Vikki found it hard to believe that anyone could experience a Fourth or Fifth Dimension and not know it, and said so. (pp. 39–40)

From point, to line, to shape, to solid, and beyond—the imaginary journeys between dimensions were indeed fascinating, eventful, and unpredictable for Albert and Vikki in Flat- and Flatterland, respectively.

Within this March issue of the *Gazette*, you will encounter three new articles and nine regular columns, covering a number of exciting dimensions of mathematics content, teaching, and research.

In *Recasting Mad-Minutes: Going Back to the Basics?*, Marc Husband and Tina Rapke share the results of a classroom-based research experiment, in which Grade 4 students were asked to complete "mad-minute" 1x2-digit multiplication questions, some (control group) using traditional algorithms, and others (treatment group) being encouraged to analyze the questions in terms of perceived difficulty level, to use a variety of previously encountered solution strategies as appropriate, and to generate and share their own set of similar questions and solutions with their peers. The results of pre- and post-tests written by each group are discussed, as are implications for teaching in light of the Ontario Curriculum.

In their article, *The E-Brock Bugs Computer Game: What If Becoming a (Better) Mathematician Was a Fun-Filled Adventure?*, co-authors Laura Broley, Chantal Buteau, and Eric Muller lead us through a bug's-eye view of the recently released (2013) digital version of an intriguing educational board game that was originally developed at Brock University back in the 1980s, and that has been successfully used to help elementary and secondary (MDM4U, in particular) students explore the major concepts of probability. Join narrator Bumble the Bee as she encounters helpful friends Bugzy and

Smarty, as well as the evil gang of Mac, Bash, Fitz, Trickz, Crazee, Wicked, and the ultimate mastermind, "Dr. P," and attempts to free the six districts of Bug City. A project website featuring the free game download, teacher resources, media coverage, and related publications is also shared.

Finally, in *Vectors and the Track and Field Jumping Events*, author Patrick Russell explains how vectors (arrows that describe direction and magnitude of an object in motion) are applied in the various track-and-field jumping events: long jump, triple jump, high jump, and pole vault.

Regular columns include the following highlights: OAME President, Paul Alves (President's Message) introduces us to Daniel Kish, *The Batman*, and elaborates on the related importance of appropriately high teacher expectations, growth mindsets, and the fostering of challenging learning opportunities for our students; Todd Romiens (OAME/NCTM Report) discusses three NCTM websites—*Illuminations, Figure This, and Reflections*; Stewart Craven (Fields Institute Report) reports on the five speakers involved in the Annual Research Day (Jan. 31, 2015) of the MathEd Forum; Mary Bourassa (Technology Corner) highlights PhET interactive simulations, and revisits the open-source software, Desmos; Shawn Godin (What's the Problem?) poses a problem regarding completing the square; Lynda Colgan (Hey, It's Elementary) takes us up, up, up, and away with her detailed description of the history and construction of the Bell tetrahedral kite; Mirela Ciobanu (In the Middle) looks at using efficient visual representations to solve mathematical word problems; Ann Kajander (MB4T) explores the importance of connecting patterns to multiple algebraic descriptions; and Greg Clarke, Agnes Grafton, Ross Isenegger, and Markus Wolski (Provincial Digital Learning Resources) share some classic Geometer's Sketchpad (GSP) files, and also provide us with an exciting update regarding their recently released Rekenrek app for desktop and mobile devices. As usual, this issue also features the rich contribution of ideas for elementary math teaching as found in the Abacus insert, co-edited by Mary Lou Kestell and Kathy Kubota-Zarivnij, and which focuses on spatial reasoning within measurement contexts.

Volume 53 Issue 3 also includes two special features: a review by Tim Sibbald of Inouye's creative new children's book, *Introductory Calculus for Infants* (2011); and, the first instalment of a series of research spotlights focusing on current projects, as shared by Ontario mathematics education researchers (note: further such submissions for future issues are welcome).

The Canadian Mathematics Education Study Group (CMESG) annual conference (more information at their new website: **www.cmesg.org/**) will be taking place at the Université de Moncton, Moncton, New Brunswick (June 5–9); and, a Fields Institute-sponsored Math + Coding Symposium (June 19–21) will be hosted by the Faculty of Education at Western University, London, Ontario (more information: **www.researchideas.ca/coding/**).

Gazette is from the early seventeenth-century Italian word gazzetta—originally Venetian *gazeta de la novità*, meaning "a half-pennyworth of news," because the news sheet sold for a *gazeta*, a Venetian coin of small value. While the *Ontario Mathematics Gazette* has moved to a predominantly digital subscription base (i.e., approximately 1900/2200, or 86 percent of OAME membership), we trust that the rich content—both the *key-processed* and *tree-processed* versions—will continue to provide a wealth of multi-dimensional stimulation for your professional reading pleasure. We look forward to receiving your future submissions.

References

Abbott, E.A. (1998). *Flatland: A romance of many dimensions.* London, UK: Penguin. (Original work published 1884)

Caplan, S., & Wallace, W. (Producers), & Johnson, D. (Director). (2007). *Flatland: The movie* [Motion picture]. United States: Flat World Productions.

Stewart, I. (2001). *Flatterland: Like Flatland, only more so.* Cambridge, MA: Perseus Books.

Vitolo, F.X. (Producer), & Ehlinger Jr., L. (Director). (2007). *Flatland: The film* [Motion picture]. United States: Self-produced. ▲

Mathematics is the art of giving the same name to different things.
Henri Poincaré

May not music be described as the mathematics of the sense, mathematics as music of the reason? The musician feels mathematics, the mathematician thinks music: music the dream, mathematics the working life.
James Joseph Sylvester

▲ President's Message

PAUL ALVES
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"Running into a pole is a drag, but never being allowed to run into a pole is a disaster." These words struck me on the way to school one day as I was listening to a podcast. The podcast was This American Life, a weekly National Public Radio show that puts stories together around a specific theme. The title of this show was Batman (Episode 544 – **www.thisamericanlife.org/**), a reference to the person who uttered those words, Daniel Kish. Although the title and story primarily focus on Daniel's remarkable story, the theme that emerges is the impact of our expectations on those around us.

When we first encounter Daniel, he is leading one of the documentarians on a hike through a trail in southern California. This seems completely unremarkable, except for the fact that Daniel is blind. He leads the hike and finds his way along the trail through the use of echolocation. Echolocation uses the echoes that are perceived by the ear from an emitted call, much like bats find their way in the night. Daniel uses clicks that he creates by using his tongue and the top of his mouth. He makes his way along the trail with ease, skirting the edge of cliffs, using a walking stick, and all the while, clicking to perceive what is around him. Daniel has gained some media notoriety for his skill, especially since he uses it to ride a bike.

The story then returns to Daniel's childhood and how he acquired the skill. Of particular interest was his mother's reaction when he lost his sight at the age of five. Daniel's mother relates how she faced a decision: What should she do now that her child was blind? How would she respond as a parent? In short, what she decided was to simply open the door and let him be a five-year-old boy. As he explored his environment, by necessity and almost by instinct, he started clicking. Once he realized that clicking provided him with the ability to make his way around his world, he did the same things that all five-year-olds do: climb a tree, run in his yard, and one day, ride a bike. He had a particular favourite game of starting from the top of his road and barrelling down full speed on his bike. One day, he didn't react quickly enough and ran into a pole. The bike and Daniel were a mess, but like any kid, he got up and, once his mother picked up a new bike, he started riding again. This is where the quotation came in and resonated with me as I related it to teaching. Sometimes, as a teacher, the problems I provide don't allow for the productive struggle, and sometimes failure, that leads to the learning. This would be that zone of proximal growth. My expectations of my students influence my instructional decisions and in turn affects student learning.

Eventually Daniel reached a point in his life where he wanted to share his skill. In the podcast, we listen in as a young boy is being taught by his blind teacher to click and explore his environment at a field near his home. His caregiver watched on as the young boy learned to trust what his ears were perceiving around him. Along one side of the field runs a busy road, and I listened with anxiety as the child approached the road and clicked to find his way. Finally, when the boy was just a few feet from the road, his caregiver reached out and pulled him back. What the boy's teacher says at this point goes to the heart of what I still struggle with in my own classroom. By reaching out and pulling the child back, the young boy's teacher gently lets the caregiver know that we have removed the learning opportunity for the student. We can't blame the caregiver for reaching out and guiding the child to safety. In the same way, when we are in the classroom and jump in when we see a student struggle, we do this from a desire to see the child succeed. But in much the same way, we have removed the opportunity to learn. Now there is a difference between allowing our children or our students to struggle with a problem and protecting them from physical harm. But the parallel drawn here is between the skill that the young boy was trying to acquire to become independent of his caregiver and how we want our students to become critical thinkers and lifelong learners. The struggle and sometimes mistakes along the way are important parts of that process of learning.

This idea of allowing students to struggle is an important stage in the learning process and forms a key part of what is called the "growth mindset." The growth mindset was introduced by Stanford professor Dr. Carol Dweck and laid out in her book *Mindset: The New Psychology of Success*. The podcast actually includes a brief interview with Dr. Dweck as she discusses the impact of expectations. The growth mindset has been a significant theme of some of the professional development that OAME has organized and delivered for teachers this school year. The Leadership Conference in November featured Dr. Jo Boaler, a leading advocate for teaching math with a growth mindset. The conference was a high point for me this year. Perhaps you were there or followed what was happening on Twitter. The demand to attend was overwhelming and we had to close registration. Amy Lin kicked off the Leadership Conference by sharing her Tales of Passion, and the conference was concluded on the Saturday, with Dr. Cathy Bruce and Shelley Yearley speaking on the visible and invisible work of math leaders.

If you didn't get a chance to make it out to the Leadership Conference, but want to participate in high-quality professional development and learn more about what teachers are doing around mindset, the provincial conference is around the corner, and mindset forms the backdrop for many of the sessions in the program. The theme for the conference is *Building Mathematical Mindsets*. The conference is being held at Humber College and will be a great opportunity to hear speakers, but to also share as a community. If you are unable to attend in person, don't forget about the opportunity to attend through the eConference. You can get more information by going to **www.oame2015.ca**.

I hope that you will get an opportunity to make it out to the conference, but if you can't, why not download the podcast and start a dialogue with your colleagues about the value of struggle in the learning process? Again, the struggle may be a drag, but it isn't a disaster. ▲

▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Jamie Pyper, Queen's University

Projects: Jamie is conducting a multi-year inquiry into pre-service teacher efficacy within the Consecutive one-year program for Primary/Junior and Intermediate/Senior Division pre-service teachers. He is also researching in the area of pre-service mathematics teacher discourse from a social semiotic and linguistic perspective. Currently underway is yet another project (with Dr. Richard Reeve, science, and Dr. Jane Chin, English) examining "professional literacies" for mathematics, science, and English pre-service teachers in the context of curriculum B.Ed. courses. Jamie serves as the Coordinator of the Mathematics, Science, and Technology Education Group (MSTE) at Queen's University, and they have just completed an e-zine, which is now available via their website (**www.educ.queensu.ca/mste**). ▲

Math Education Researcher: Dr. George Gadanidis, Western University

Projects: George has been involved in many collaborative projects including: Random Acts of Math (**www.researchideas.ca/randomacts**); Mathematics + Coding (**www.mathNcode.com**); Math Classroom Documentaries (**www.researchideas.ca**); Math Courses for Teachers (**www.researchideas.ca/domath**); Math Concerts (**www.researchideas.ca/jx**); Math Performance Festival (**www.mathfest.ca**); Interviews with Mathematicians (**www.fields.utoronto.ca/mathwindows**); and, What will you do in math today? (coming soon @ **www.researchideas.ca/wmt**). ▲

Math Education Researcher: Dr. Ann Kajander, Lakehead University

Projects: Ann is continuing to explore the impact of specialized mathematics courses on the development of elementary pre-service teachers in mathematics methods courses during teacher education. Her new co-authored (with classroom teacher colleague Tom Boland) book, *Mathematical Models for Teaching: Reasoning Without Memorization* (2014), draws upon this ongoing work. ▲

▲ Technology Corner – PhET Interactive Simulations and More from Desmos

MARY BOURASSA
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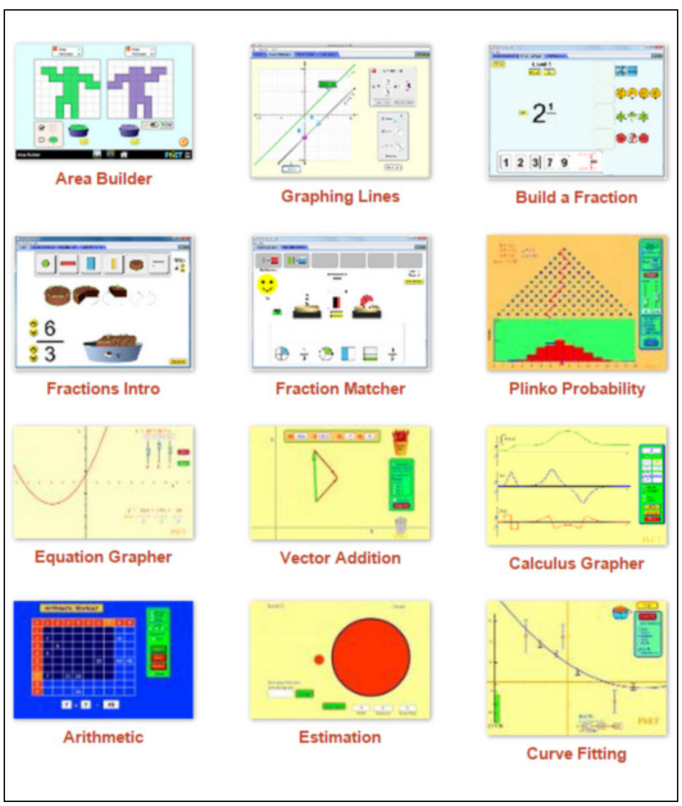
TWITTER HANDLE: @MaryBourassa

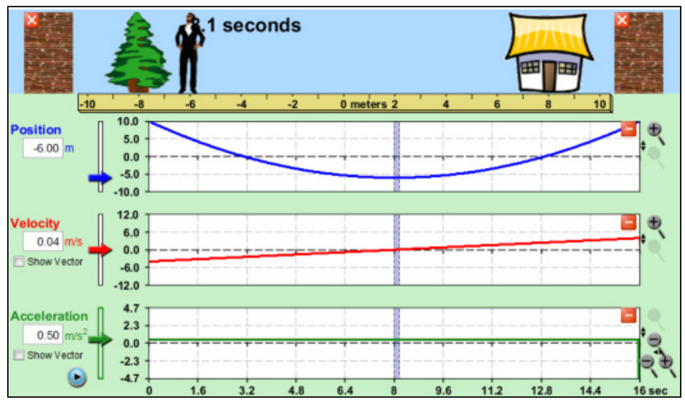
Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

PhET

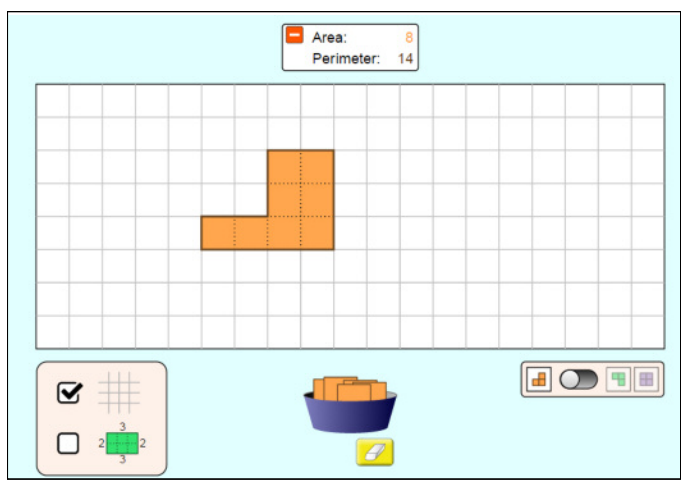
PhET (**phet.colorado.edu/en/simulation/moving-man**) is a collection of free interactive simulations from a group out of University of Colorado, Boulder. Some of the many different simulations for math and science are shown below.



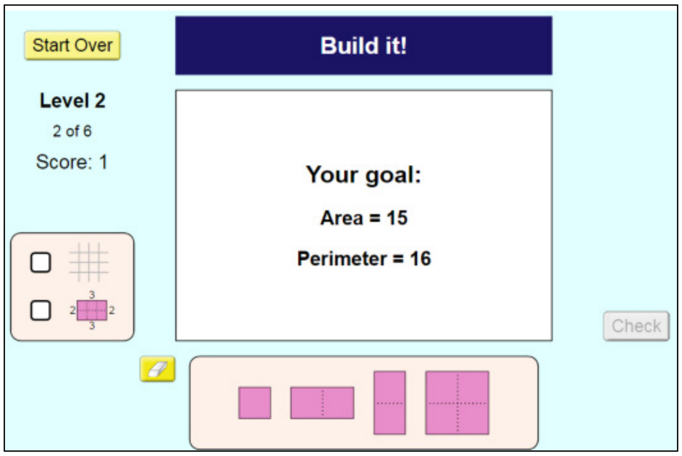
My favourite simulation is "The Moving Man," which I have used from Grade 9 to calculus. The man in the simulation can start anywhere along the horizontal axis from -10 to 10, and you can set the velocity and acceleration. Your students can predict what graphs will look like and explore many "what if" questions. This is a great compliment to working with a motion sensor to explore relationships in Grade 9. In calculus, students can easily see the graphs of the first and second derivatives of the position graph, and make connections between inflection points, maximum/minimum points, and zeros. It also really helps students understand the meaning of positive and negative velocity and acceleration.

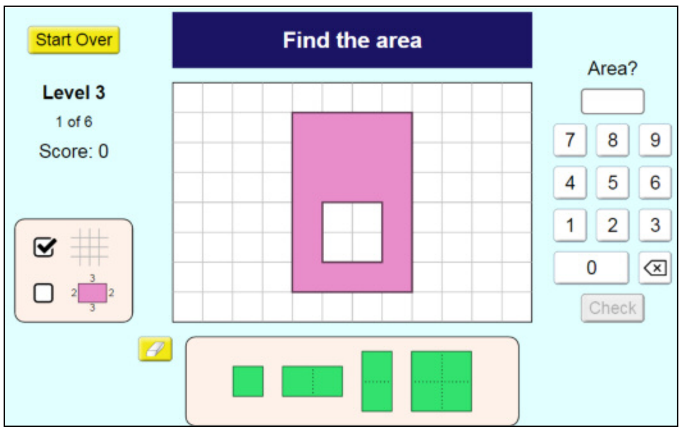


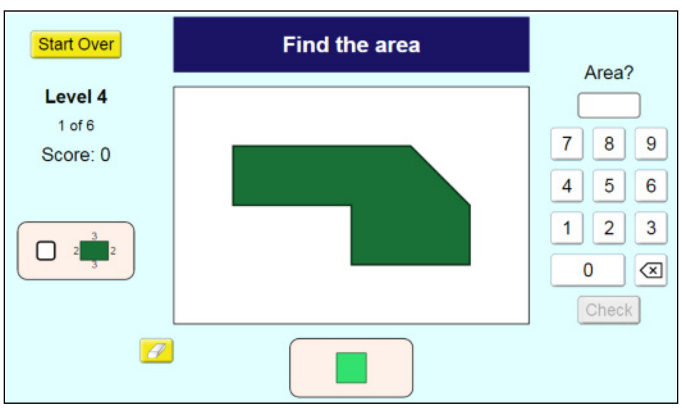
Another simulation from PhET is called "Area Builder" (which is also among the HTML5 simulations). It allows students to create their own shapes, with options to see the grid and the side lengths. They can then explore the relationship between perimeter and area of different shapes, and can compare between two shapes.

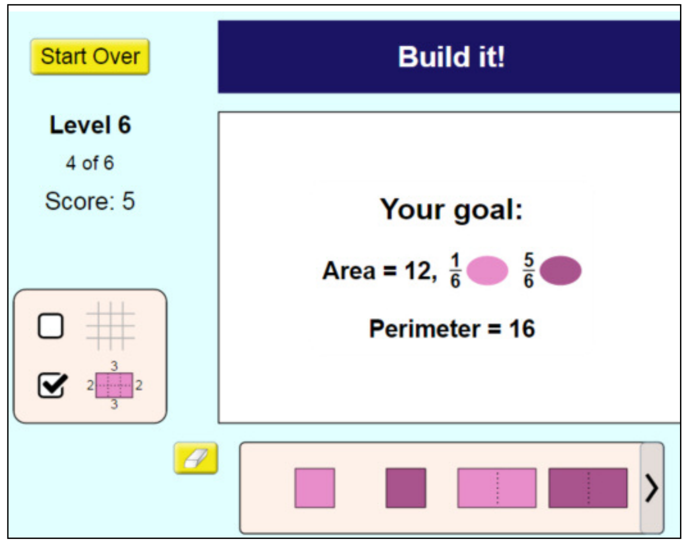


There is a game included which has students progress from building shapes with a specific area to finding the area of composite shapes to building a shape with a specific area and perimeter and a given ratio of one colour of blocks to another colour of blocks. This provides students with the opportunity to progress at their own pace, and the higher levels definitely provide a challenge.







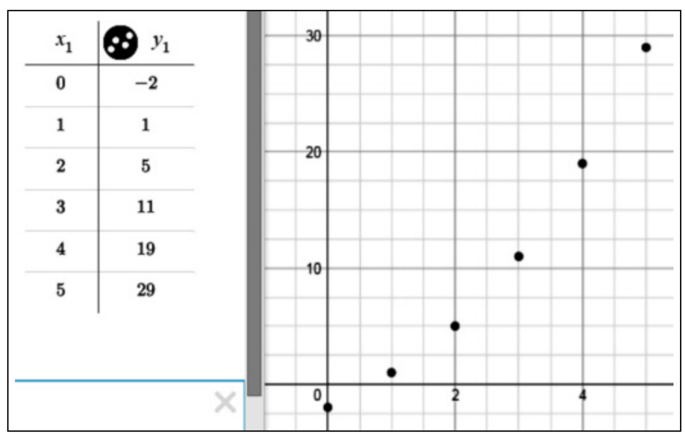


There are links to teacher resources posted on each simulation page, including ones with ready-to-go handouts.

Desmos

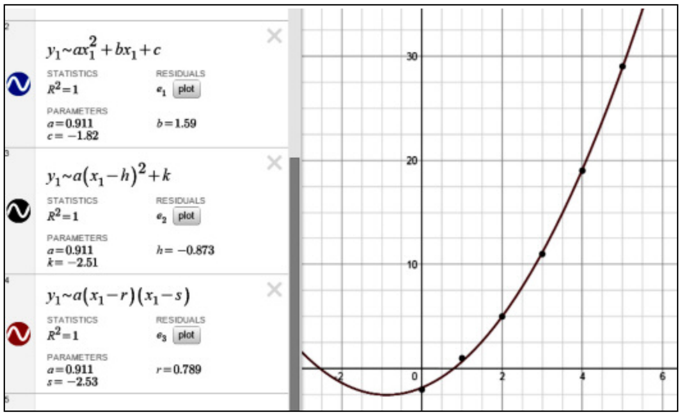
Desmos (**www.desmos.com/**), the free graphing software/app, continues to support teachers by adding functionality to its product and by creating great class activities.

Desmos is now capable of doing regression and it works very nicely. Here is some sample data:

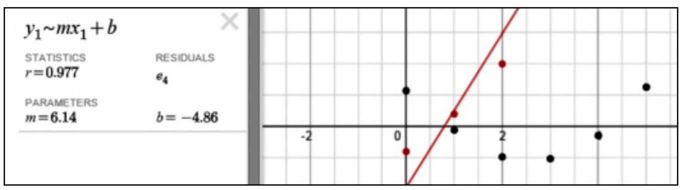


x_1	y_1
0	−2
1	1
2	5
3	11
4	19
5	29

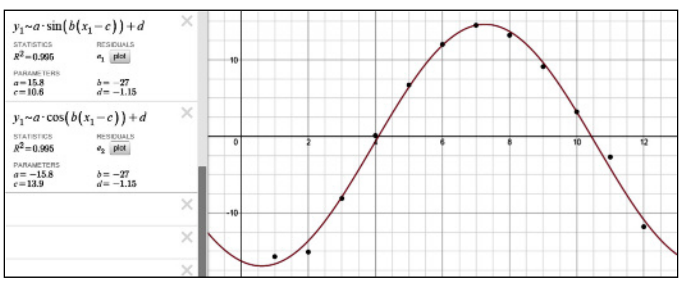
Note that the columns are labelled with x_1 and y_1. To have Desmos perform a regression, enter the equation using x_1 and y_1 as variables and replace the equal sign (=) with a tilde (~). What I really love is that I can enter a quadratic equation in any form I like and Desmos will do the work. Here are the results for quadratic regressions on this data:



The R^2 value is shown in each case, and you can plot the residuals. If you had chosen the wrong model for your regression, then opted to plot the residuals, they would form a pattern, as seen here, with a linear regression on the above data:



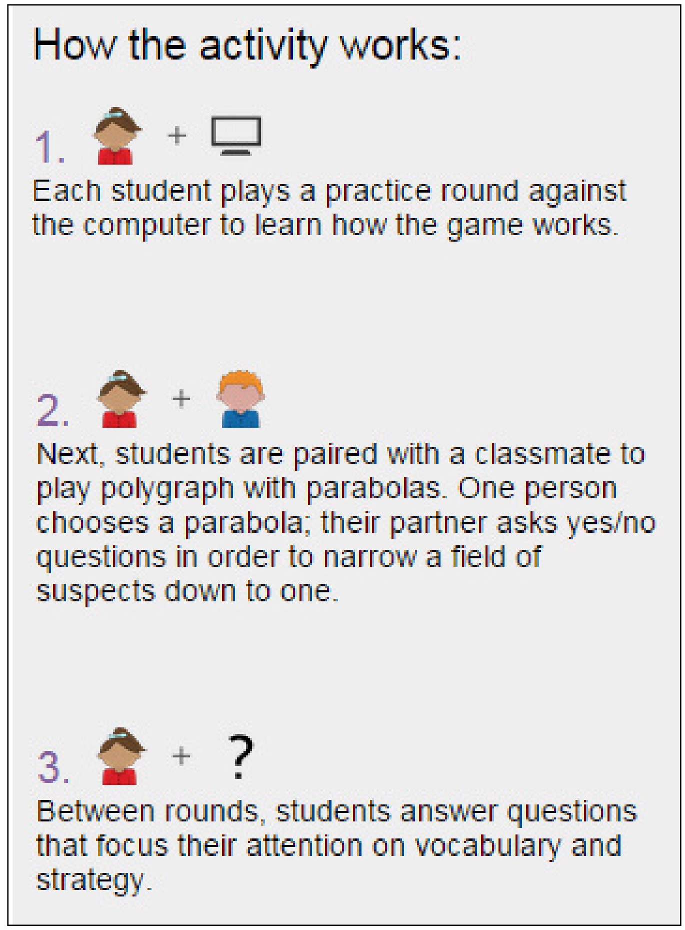
Here is another example of a regression on temperature data. Note that you can do a sine or cosine regression.



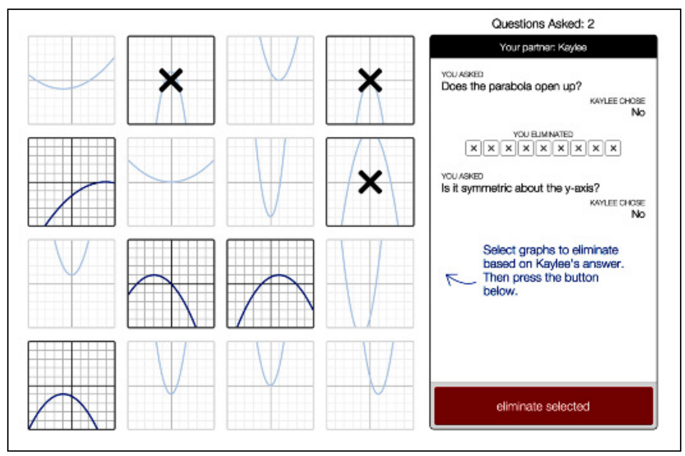
This is a very powerful tool worth exploring with your students.

Following the success of Penny Circle, Desman, Function Carnival, Water Line and Central Park, Desmos released a new class activity in December called Polygraph. There are four versions of Polygraph: Parabolas, Lines, Hexagons, and Rational Functions.

These are like the “Guess Who?” game, where each player is matched up with another player from his or her class. One student chooses a graph, and his or her partner must ask questions that have a yes/no answer in order to discover the chosen graph.



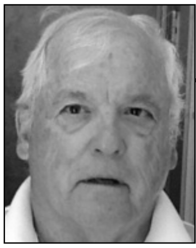
Here is a snapshot of the parabola game in action:



As the teacher, you can see all the games in progress and what questions are being asked. It is a fun way to have students focus on using the correct vocabulary and purposely use what they know about the features of a parabola.

All of these activities can be found at **teacher.desmos.com/**. Follow **@Desmos** on Twitter to be the first to hear about the latest releases. ▲

▲ OAME/NCTM Report



TODD ROMIENS
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Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette.

What we are doing as educators in classrooms today is vastly different from what we were doing a few years ago. Over the past number of years, the understanding of how students learn math has greatly improved. At all grade levels, we actively engage students in the learning process. Students are challenged to think more deeply about the problems they are solving, and are encouraged to make connections to real-life situations. The use of technology has opened many doors to students that perhaps were previously unattainable.

NCTM offers many electronic resources to facilitate the learning process. They are generally free and accessible on the NCTM website **(www.nctm.org**). The following three resources are of particular interest:

The "Illuminations" website (**www.illuminations.nctm.org**) contains classroom-ready lessons for teachers of pre-K–12, as well as interactive tools that help bring math concepts to life in class. Reviews of online resources round out the collection. My last column contained two examples.

The "Figure This" website (**www.figurethis.org/index.html**) provides 80 math challenges for middle school students. Each challenge explains how the math is used in the real world, and provides a hint to get started, complete solutions, questions to think about, and resources for further exploration. Designed to help families do math at home with their children, the site offers materials to help families support their children's math studies.

One example of material from the "Figure This" website is a suggestion to help parents when they ask, "How come the math my child brings home doesn't look like the math I remember?" The website's response is as follows:

If you don't recognize the math in your child's homework, think about how the world has changed since you were in school. The math books are different because the world is different.

The basics are changing. Arithmetic skills, although important are not enough. To succeed in tomorrow's world, students must understand algebra, geometry, statistics, and probability. Business and industry demand workers who can:

- solve real-world problems
- explain their thinking to others
- identify and analyze trends in data
- use modern technology

The mathematics students do in middle school should prepare them for the new basic skills necessary for their future. Instead of worksheets, your child may bring home problems to investigate that are related to real life.

Giving students opportunities to learn real math maximizes their future options.

The "Reflections" website (**www.nctm.org/resources/content.aspx?menu_id=598&id=6372**) is designed to enhance the ability to critically reflect on teaching methods, by watching videos of other teachers. The site assists educators in looking at math teaching through the examination of lessons and sample student work. Lessons on algebra in Grades 3–8 are provided.

When you access the "Reflections" website, you will be prompted to choose "Reflect by Lesson" or "Reflect by Topic." A variety of Grades 3–8 lessons are available, ranging from Grade 3 patterns to Grade 8 bridges. When you click on the one you are interested in, you will access the lesson, a lesson plan, some student work, and some problems that are designed to engage your students. You will need Windows Media Player in order to view the videos. In reviewing this material, you will find additional detail in the lesson plan, including a possible procedure to follow, materials needed, and classroom set-up to maximize student participation.

After you have selected a lesson to teach, go to "Reflections by Topic," which will provide access to detailed work associated with the lesson, such as activity sheets for students, examples of how students have responded to questions, and assessment strategies. Also included is a set of questions for educators to consider as they probe deeper into the methods used to teach the lesson.

Please be reminded that the NCTM annual conference is in Boston, Massachusetts from April 15–18, 2015. ▲

▲ What's the Problem? Completing the Square

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Shawn is the head of Mathematics, Business Studies, and Library at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors and is currently Editor-in-Chief of the Canadian Mathematical Society's problem-solving journal, Crux Mathematicorum*.*

Welcome back, problem solvers. Last time, I left you with the following problem: The point A(0, 0) is the vertex of a square ABCD, and the point M(5, 0) is the midpoint of side BC. Determine the coordinates of vertices B, C, and D.

This problem was inspired by question 32 from the problem-solving journal, *Crux Mathematicorum*. The original problem appeared in volume 1, issue 4 from June 1975, and the solution appeared in volume 1, issue 7 from September 1975.

This is a nice problem, as it allows us to broach it in a number of different ways. Let's start by using an 11 by 11 peg geoboard.

If we consider the middle of the left side to be A(0, 0), then the centre pin will be M(5, 0), and we can experiment with positions for B. If B is in the correct position, then there will be a right angle at B, and AB will be twice as long as BM. We can check our answer by measuring lengths and angles on the geoboard.

Playing with the position of B, we can make several discoveries. For example, if the *x* coordinate of B is left constant and we vary the *y* coordinate, we discover that as *B* gets closer to the *x* axis, the angle increases, while *AB* and *BM* decrease. Figure 1 illustrates that, $\angle B_2 > \angle B_1$, $AB_2 < AB_1$ and $B_2M < B_1M$.

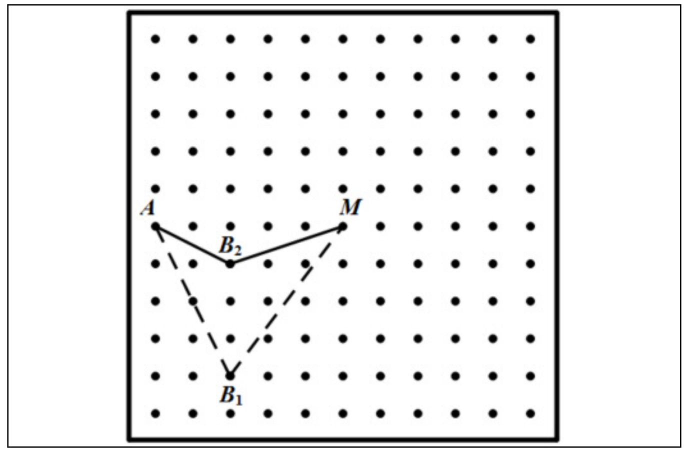

Figure 1

Similarly, if the *y* coordinate of B is left constant and we vary the *x* coordinate, we discover that as B gets closer to the *y* axis, AB decreases, while BM increases. It is clear in Figure 2 that $AB_2 < AB_1$ and $B_2M < B_1M$.

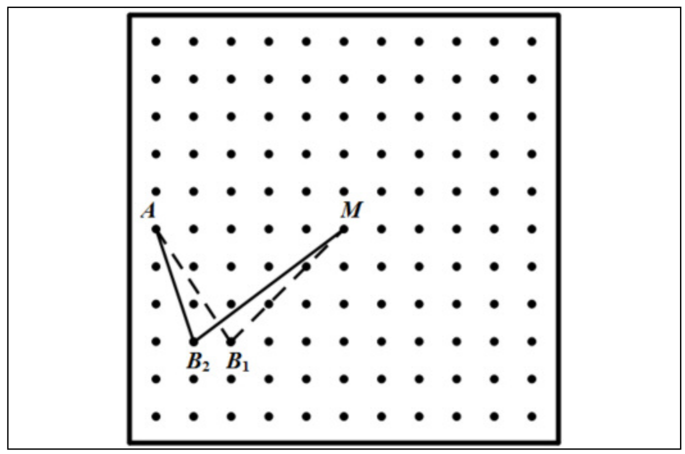

Figure 2

The behaviour of the angle in Figure 2 is more complex. When B is on the perpendicular bisector of AM, which we cannot do with the geoboard since it is the line with equation $x = 2.5$, the angle is maximized. As we move further "from the centre," the angle decreases. Figure 2 also shows that, although it is difficult to see without measuring, $\angle B_2 \approx 71^\circ < \angle B_1 \approx 78^\circ$.

With these discoveries, or through trial and error, we find that if B is placed at (4, –2) or (4, 2), the conditions of the problem are satisfied. The other vertices can then be easily determined.

We could have gone through a similar process using dynamic geometry software. We can also use dynamic geometry to solve the problem, using some geometric properties. For example, a triangle inscribed in a semi circle is always a right triangle. Thus, if we construct a circle with AM as a diameter, then B must fall on this circle.

Next, B must also be twice as far from A as it is from M, so let's do a little experimenting. Draw a circle with centre A and measure its radius. Using the calculator tool, divide the radius of the circle by 2, and use the result as the radius of a circle with centre M. We can adjust the radius of the first circle until the points of intersection of the circles centred on A and M lie on the circle with diameter AM. This is point B such that $\angle ABM = 90^\circ$ and AB = 2BM. Figure 3 shows that this occurs at the two points discussed previously.

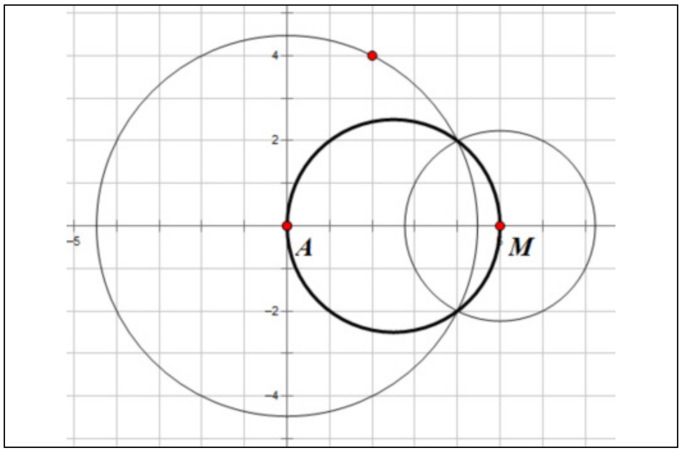

Figure 3

If we experimented further with our sketch, we may discover something else. If we constructed the locus of points, P, that satisfies AP = 2PM, we would make an interesting discovery. This locus of points is a circle! This is known as the Apollonius circle theorem. It states that for two points, A and B, and a positive integer, $r \neq 1$, then the locus of points, P, that satisfies $\frac{AP}{PB} = r$ is a circle. If $r = 1$ then the locus is the perpendicular bisector of segment AB, which can be thought of as a circle with infinite radius, with centre at infinity.

Using the Apollonius circle theorem, (**en.wikipedia.org/wiki/Apollonius%27_theorem**) since we want points such that AP = 2PM, we can find two of them very quickly. Obviously, the point P(10, 0) is one such point, since M is the midpoint of AP. Also, if we find the image of A under a dilation of factor $\frac{1}{3}$ centred at M, then this point $Q\left(\frac{10}{3},0\right)$ also satisfies the condition. Thus, if we construct a circle with PQ as a diameter, then B is located at either intersection of this circle and the circle with diameter AM. Figure 4 shows the two circles with one of the possible squares.

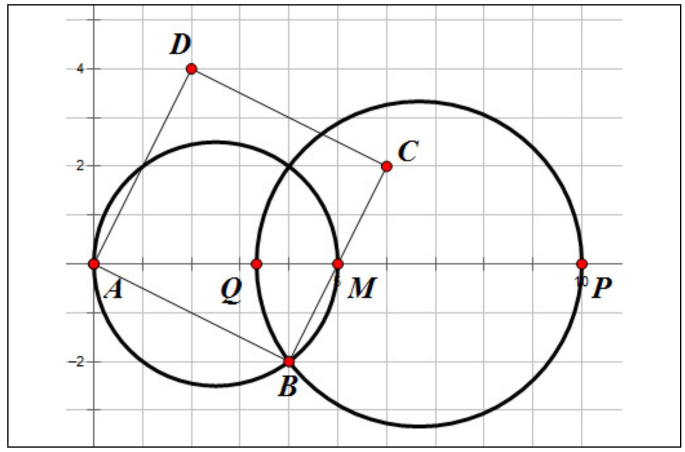

Figure 4

Let's look at one more solution that uses ideas from the Grade 10 academic mathematics curriculum. From the conditions of the problem, triangle ABM is right angled, with right angle at B and AB = 2BM, which is shown in Figure 5.

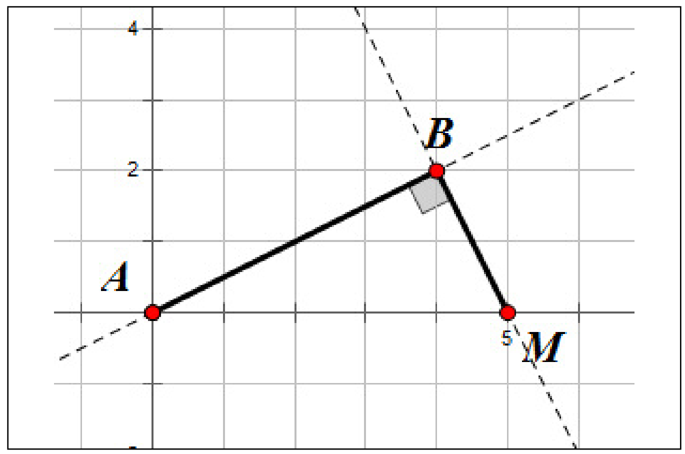

Figure 5

If we look at one possible location for B, then we must have

$$\tan A = \frac{BM}{AM} = \frac{1}{2}$$

We also know that the slope of a line is equal to the tangent of the angle that it makes with the positive x axis. Thus, AB is contained on a line through the origin, with slope $\frac{1}{2}$, that is the line with equation $y = \frac{1}{2}x$. We also know that BM is perpendicular to AM, thus BM is contained on a line through M, with slope –2, that is the line with equation $y = -2x + 10$. The location of B is the point of intersection of the two lines, which we can get by solving a system of equations.

The last solution, and the solution using the Apollonius circle theorem, could be used to see that a solution exists, and our methods work, if the problem were reworded as: *The point A(0, 0) is the vertex of a square ABCD, and the point M(5, 0) is on side BC such at* $\frac{BM}{AM} = r$ *for some rational number* $0 < r < 1$*. Determine the coordinates of vertices B, C, and D.* We could solve the problem with *r* being irrational, using algebra, but in the case of rational *r*, you can also solve it geometrically (i.e., constructing the circles, or drawing the lines with specific slopes). You may want to amuse yourself with the following variant. Keep A at the origin, and put M at (10, 0). This problem has nice solutions, that is, it can be solved using all methods from this article, for the case where M is the midpoint and the case where slope $= \frac{1}{3}$. Have fun!

Now for your homework:

You have two hourglasses: one that measures 9 minutes and one that measures 13 minutes. Determine how to measure 30 minutes, using these two hourglasses.

Until next time, happy problem solving! ▲

▲ Hey, It's Elementary – Up! Up! Up! Taking Flight with the Bell Tetrahedral Kite

LYNDA COLGAN
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Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

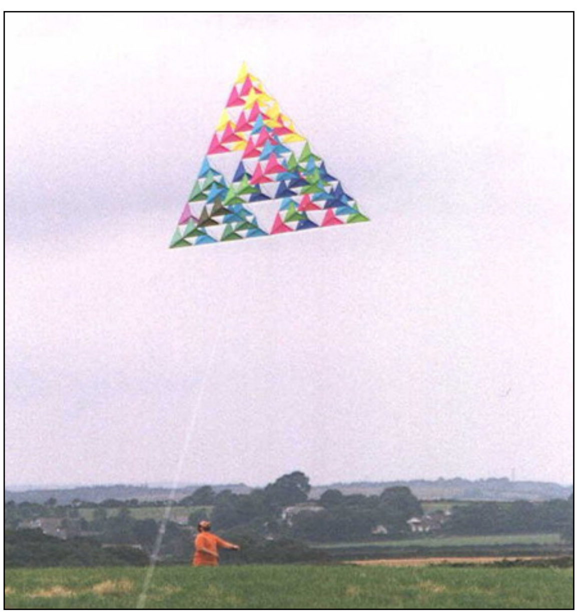
> "He goes up there on the side of the hill on sunny afternoons and with a lot of thing-ma-jigs fools away the whole blessed day, flying kites, mind you. He sets up a blackboard and puts down figures about these kites and queer machines he keeps bobbing around in the sky. Dozens of them he has, all kinds of queer shapes, and the kites are but poor things God knows! I could make better myself." *Local Boatman, commenting on Alexander Graham Bell's passion for kite flying Baddeck, Nova Scotia, 1860*

The conditions in Kingston, located on the shore of the St. Lawrence River, are often perfect for celebrating "windsdays." The first "windsday" was celebrated in 1968 with the release of the Disney movie *Winnie the Pooh and the Blustery Day*. The story begins one blustery day, when Winnie the Pooh travels to his favourite spot—where he "sits and thinks"—when suddenly his friend, Gopher, surfaces and says he's "skedaddlin'" away because it was *windsday*! Not understanding Gopher's dire message, Pooh sets off to wish all his friends a happy *windsday*, starting with his best friend, Piglet. When Pooh arrives at Piglet's house, he finds his friend having a most difficult time gathering up all the leaves that were blowing into his yard. Suddenly (and much to Pooh's horror), Piglet is whooshed off the ground in an updraft—flying further and higher away into the sky. At the last possible second, Pooh grabs hold of Piglet's long knitted scarf, which luckily unravels into one long strand of wool. As Piglet clings onto the last bit of his scarf, it looks as if Pooh is flying Piglet like a kite—and so was borne the tradition of celebrating "*windsdays*" by flying kites.

Although I have no hard data to substantiate my claim, I believe that the Fort Henry Hill has to be one of *the* best sites in the world for kite flying. On many glorious sunny days, when gusty breezes rise from the river, it is hard to believe that anyone would ever want to fly a kite anywhere else, and clearly, I am not alone in this belief: predictably, in such conditions, an horizon full of colourful and energetic kites of all sizes and shapes can be seen making the point gracefully and beautifully, performing joyful airborne dances in the sky over town.

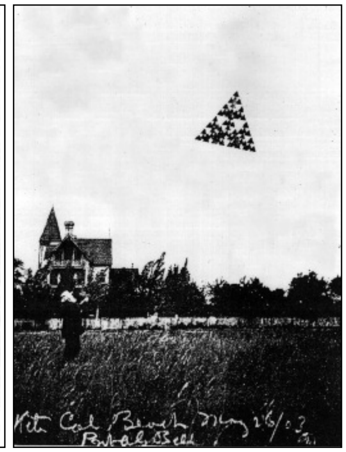
As a child, my Dad often took me to Ashbridge's Bay in Toronto to fly kites. Although it was fun to spend time with my father, I don't remember my kite often catching the wind or taking wing. I do have mental pictures of me running in one direction and looking back in the other in disappointment as my kite took yet another nose-dive into the wet sand at Lake Ontario's edge. Putting *windsdays* behind me, I moved on to other hobbies.

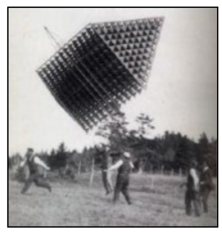
Fast-forward several decades: kites once again made their beautiful presence felt in my life. Nick, my "brother" by luck, not blood, was a competitive kite flyer. Over the years, I watched him participate in these amazing spectacles of ballet and precision in the sky. As the third anniversary of his untimely death from an aortic aneurism approaches, I offer this column as a tribute to Nick. May one mathemagical creation—a tetrahedral kite—inspire you and your students to fly your beautiful creations, and your imaginations, atop local hills.



The tetrahedral kite featured here has a long and rich tradition. It is based on the work of Alexander Graham Bell, who used kites first to test out his ideas about flight without risk to human life. Bell tested his tetrahedral kites in different configurations, elevations, pulling them by horse or boat.

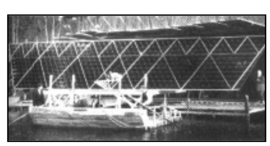
Alexander Graham Bell speaks during the first public demonstration of tetrahedral kites in the spring of 1904 in Baddeck, N.S.
Parks Canada, Alexander Graham Bell National Historic Site/ Canadian Press



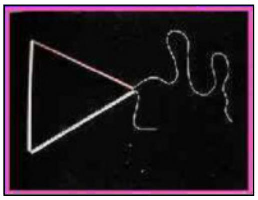


The kite shown in the photograph here had 1300 tetrahedral cells and 440 square feet of lifting surface. On Christmas Day 1905, it accidentally carried a coachman's brother, Neil MacDermid, 30 feet into the air. This happy accident confirmed Bell's theory that a tetrahedral structure could be designed to carry both man and motor—a safe and stable tetrahedral airplane.

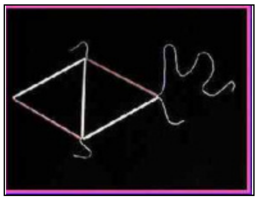
In 1907, Bell built another tetrahedral kite, which he called the "Cygnet," consisting of 3393 cells, with an opening in the centre to carry a man. Equipped with floats and towed by a steamer, it flew beautifully for 7 minutes, reaching a height of 163 feet over Baddock Bay, Nova Scotia, carrying Thomas Selfridge of the U.S. Army. The descent was so gentle that, with his vision obscured by the structure around him, he was unaware that he was coming down, until the kite touched the water. Unfortunately, the line from the steamer was not released quickly enough, and the kite was destroyed as it was dragged through the sea.



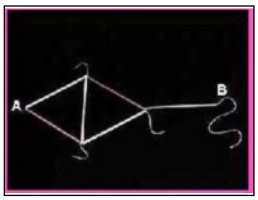
To make a simple version of Alexander Graham Bell's tetrahedral kite, you will need 24 eight-inch straws (not the bendable kind), a glue stick, brightly coloured tissue paper (Bell, himself, preferred red silk), cardboard (a cereal box face will work), scissors, a ruler, pencil, protractor, and some string. To get started, cut four pieces of string about 114 cm long and four other pieces about 63 cm long.



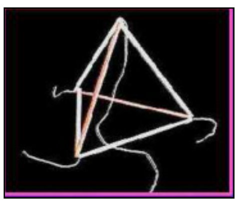
Thread three straws onto one 114 cm string and tie them into a triangle.



Place two straws on a 63 cm string and tie them to the corners of the triangle that does not have the original knot.

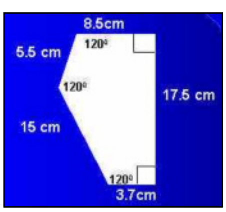


Place one straw on the long end of the original 114 cm string and tie it to the free vertex of the other triangle, i.e., connect A to B.



You should now have a tetrahedron. Make three more.

Next, begin with a full sheet of tissue paper (50 cm by 76 cm) and fold it in half four times. Your final folded tissue should measure about 12 cm by 19 cm.



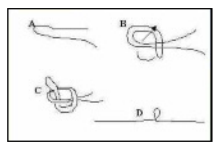
Cut this template from cardboard.

Place the template on the folded corner of the tissue paper and cut where indicated.

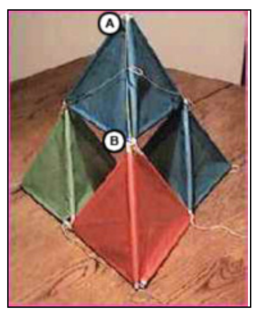
You should now have four tissue-paper pieces that look like this when unfolded. This is enough for one kite. If you want a multi-coloured kite, fold and cut more tissue.

Glue the tissue paper to two faces of the tetrahedron.

Using the strings at each vertex, tie the four tetrahedra together to make a larger tetrahedron, making sure that all the tissue faces are facing in the same direction.

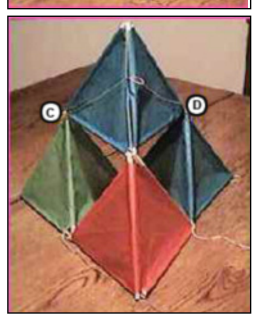


To attach the bridle to your kite, cut a loop of string about 45 cm long and tie a loop knot at its centre.



Tie one end of this string to the top of your kite at point A. Tie the other end to point B, trying to get the loop centred between the two points.

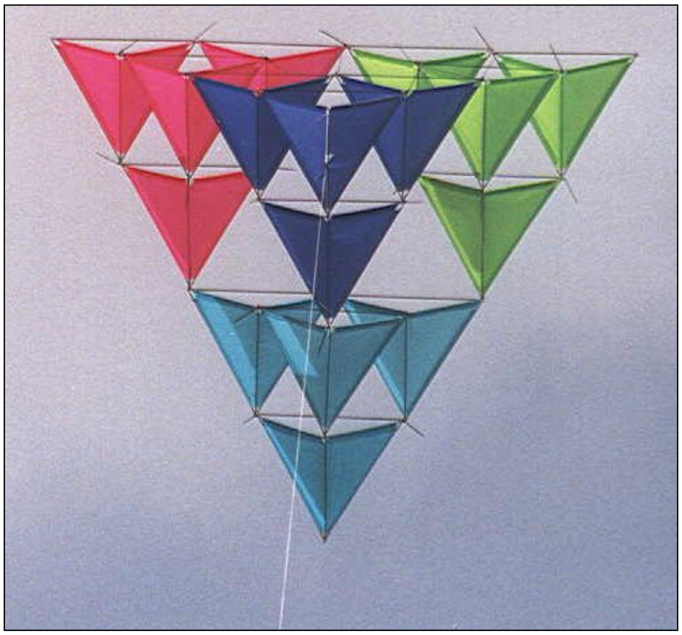
Cut a second length of string about 60 cm long. Tie one end of the string to the kite at point C.



Thread the other end through the loop knot and tie at point D. Make certain that the loop is pulled out taut before tying at D. All strings should be taut when pulling out on the loop knot.

Attach your flying string to the loop knot. Set the kite on the ground with the bridle up. Walk away from the kite about 15 m into the wind, letting out flying string as you go. If there is sufficient wind, a tug on the line will cause the kite to take off and soar nearly vertically into the air.

If you are feeling very adventurous, put four kites together to create a super-tetrahedron with 16 cells, as shown below. With attention to detail around attaching the loop knot and flying string, you should find that this kite flies even better than its smaller version—in fact, the bigger the better. The first tetrahedral kite photo shows four 16-cell kites joined together to form a 64-cell giant.



Nick D'Alto, a kite hobbyist who has duplicated Bell's tetrahedral kits, using the original patents and drawings, says, "It's a thrill to see a tetra zoom into the air and hear the weird hum as the kite strains against its line. Spectators are always surprised to see one, and they're even more surprised to learn a new side of Alexander Graham Bell. I always tell spectators that it's a Bell kite—the kite that proved large aircraft weren't impossible after all."

The sky's the limit for your tetrahedral kite—mathemagic in flight. ▲

▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

▲ In The Middle – Using Efficient Visual Representations to Solve Mathematical Word Problems

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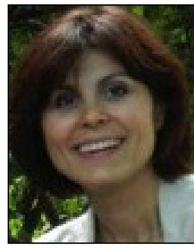


Mirela Ciobanu is currently a middle school teacher in Toronto District School Board. Mirela is a keen advocate of classroom inquiry, particularly in the area of formative assessment. Her recent work includes conducting an action research in mathematics as part of the Teacher Learning and Leadership Program for Experienced Teachers developed and funded by the Ontario Ministry of Education in partnership with the Ontario Teachers' Federation.

> "Changing representation register is the threshold of mathematical comprehension for learners at each stage of curriculum. It depends on coordination of several representation registers and it is only in mathematics that such a register coordination is strongly needed."
>
> ~ *Raymond Duval*[1]

Introduction

Classroom practice observations and recent EQAO results indicate that comprehending and solving multi-step word problems remain the biggest educational challenges for middle school students and an ongoing professional area of focus for educators. What is so problematic about solving word problems for our students? It is by far not a simple question to answer.

The framework for my analysis is Raymond Duval's research (2006) about two transformations of symbolic representations that are usually employed in mathematical processes: *treatment* and *conversion*. They play a big part in the difficulties students have with solving multi-step word problems.

First, I will use the notion of representation preferred by Duval, which is "something that stands for something

1 *Raymond Duval is Professor Emeritus at Université du Littoral Côte d'Opale (Laboratoire Mutations des Systèmes Éducatifs), France. He is known for his theory of registers (2000, 2006) in relationship with mathematical comprehension.*

else." The nature of the representation indicates the register. To clarify, I have chosen the following Grade 6 EQAO sample problem from Spring 2012 (**www.eqao.com**).

A club has money for a trip. The expenses for the trip are shown below:

- *Lunches:* $\frac{1}{4}$ *of the money*
- *Tickets:* $\frac{2}{5}$ *the money*
- *Snacks: 0.12 of the money*
- *Transportation: 20% of the money*

What fraction of the money is left over?

Show your work.

As is, this problem is encoded in both a *discursive* (words) register and a *symbolic* one (fraction, decimal, and percentage notations). The phrase "money for the trip" indicates the whole, and the word "fraction" in the question indicates the answer choice of representation register.

The *symbolic* representations are those of various quantities of the "expenses for the trip" written as various numeric notations, all parts of the same whole. Comprehension in mathematics, according to Duval (p. 115), comes from the coordination of *at least two registers of representation*. How does this ability develop in students? How much experience do they have solving problems encoded in multiple registers and not only one?

Treatments (Duval, p. 111) are transformations of representations that happen within the same register. For example, in the problem above, a student might represent $\frac{1}{4}=\frac{25}{100}$ and $\frac{2}{5}=\frac{40}{100}$ to be able to subtract the amount from the whole, $\frac{100}{100}$.

Conversions (p. 112) involve changing the register without changing the object denoted. For example, in the EQAO problem above, a student needs to convert the phrase "money for a trip" (discursive register) into $1, \frac{1}{1}, \frac{100}{100}$ or 100%. Furthermore, conversion also means that the student has succeeded in identifying the relationships between the fractional, decimal, and percentage amounts to the whole as being identical to the representation of the same content, using a mathematical equation.

For example, $1=\frac{1}{4}+\frac{2}{5}+0.12+20\%+$ fraction of money left.

The treatment chosen is obvious when using fractions or decimals or percent, but the conversion is the most complex representation transformation, one that indicates conceptual understanding. This can explain why our students usually score higher on EQAO tasks that involve treatments of representations rather than conversions, which is usually the type of processes involved in open-response multi-step questions.

Duval advises teachers to place students in problem-solving situations similar to the one I chose to exemplify his theory. Thus, such problems use various notation systems for quantities (words, fractions, decimals, and percentages) and provide rich opportunities for digging deep into their cognitive ability to convert the different representations of the same concept (e.g., the money for the trip = 1 = 100% = 1.00 = $\frac{1}{1}=\frac{100}{100}$ *or drawing a fraction area model or a 100-space grid*). The students are left to translate this problem into various notation systems and encode the answer in the required register or notation system, in this case, the fractional notation.

We must give students opportunities to convert various representations and work out the relationships between mathematical content encoded in various registers in a word problem.

The Role of Using Efficient Visual Representations to Solve Word Problems. What Can Be Learned from Singapore's Model Drawing Approach?

Dr. Richard Barwell (2011) noted that when solving word problem, students do pay attention to the three dimensions of word problems: mathematical structure, genre, and personal experience. The challenge comes, as Duval (2006) remarks, from the fact that students have a tendency to stay within mono-functional registers, even when multifunctional ones are given and treatments take the form of algorithms.

Solving a word problem involves reading-comprehension skills and writing for a purpose, using specific registers (sometimes both symbolic and discursive) and keeping the audience in mind. Thus, the purpose is economy, efficiency, and clarity of communication when writing the solution plan and answering the question. Students need to be able to recognize important information in a word problem, translate it from the discursive register into a visual and/or symbolic one, and test it back in the text of the problem. This process is not a linear one, as it involves alternating between discursive and visual, visual–symbolic, and back to discursive.

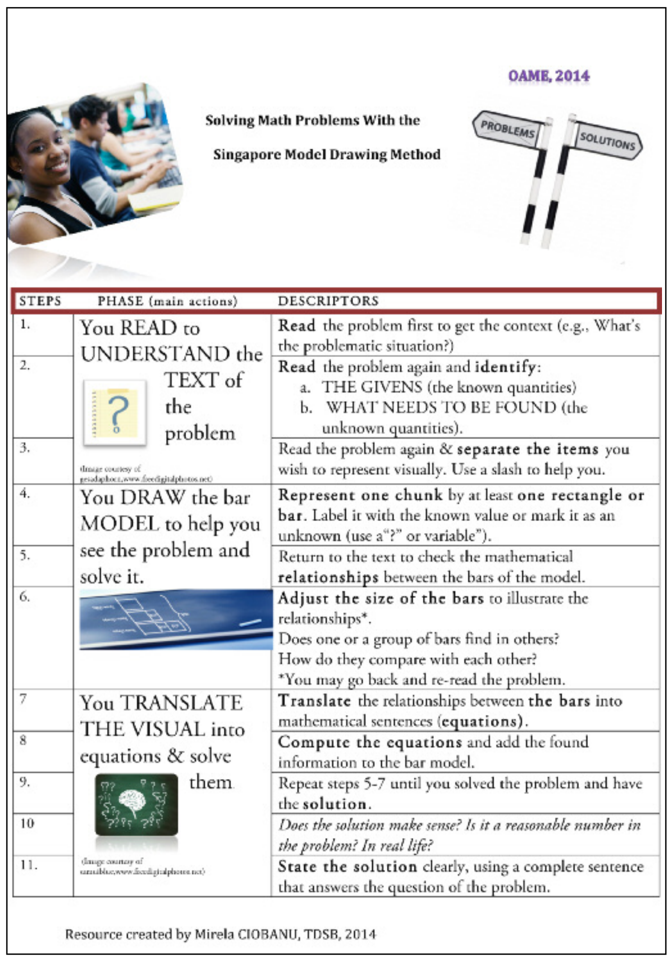
The visual model I am suggesting here is not just any "picture," but one that facilitates the *coordination of the registers of representation*, as Duval mentioned, within the same problem. We have often seen students employ a lot of visual representations that are just single object-based representations of a component of a problem, and not the relationship among all its elements, both the known and the unknown. An effective model to solve a problem is one that enables students to "*see*" the problem and have a "basis of the construction of the solution plan" (Hegarty, cited in Sajadi et al., 2013). Duval (2006) and Hegarty (1995) advise educators to avoid *shortcut* approaches to understanding problems that focus on translating key phrases into a set of computations (e.g., "more" means addition, "less" means subtraction). In other words, simply highlighting or underlining keywords or numerical information does not suffice. Using organizational tools, like KWC charts, that are meant to encourage students to identify the known, unknown, and special conditions does not necessarily mean that students understood the relationship between them. A tool like this only separates the information, but does not represent the problem structure, the relationship between its elements. In other words, it does not transform the representation of the problem into one meant to help the student understand its "problematic" situation.

Instead, we should employ a *meaningful approach* that involves translating the word problem into a model that allows the problem solver to first work out the relationships among all the elements in the situation of the problem and decide how to coordinate the registers in which the problem is encoded.

Singapore Model Drawing relies heavily on the work of educational psychologist and researcher, Jerome Bruner (1966), who believed that knowledge-representation ability is one of the factors of cognitive development. The following three modes of representation do not represent separate stages, but rather, modes of representations that are translated into each other: *enactive representation* (concrete or action based), *iconic representation* (pictorial or image based), and symbolic representation (abstract or language based). (**www.mathsnoproblem.co.uk/singapore-maths**)

The progression concrete–pictorial–abstract is not new to us. The middle stage is often overlooked, particularly in middle school, and the students experience a lot of difficulty trying to move from concrete representations to abstract ones without a pictorial stage to bridge them. Often, students rely on inefficient models of representation when encouraged to use all three registers (the famous "pictures, numbers, and words") and they do so overlapping them. Thus, they repeat the same information while communicating the solution plan, rather than switching the registers efficiently and conveniently, according to the stage of the problem-solving plan.

Singapore Model Drawing was introduced in 1983 in the country's primary school curriculum. It involves creating a visual representation model, a diagram, using rectangular bars to represent the information items in the problem. The same diagram is used not only to visualize or "see" the problem students solve, but also working with it helps them solve the problem. The Model Drawing approach is most efficient when working with proportional relationships (percents, rates, ratios, fractions, decimals). The structure of the model follows three distinct stages (Figure 1) that are interrelated, and between which the problem solver alternates while carrying the plan to solve the problem. The model was adapted by me and based on the work of Ng and Lee (2009).


OAME, 2014

Solving Math Problems With the Singapore Model Drawing Method

STEPS	PHASE (main actions)	DESCRIPTORS
1.	You READ to UNDERSTAND the TEXT of the problem	**Read** the problem first to get the context (e.g., What's the problematic situation?)
2.		**Read** the problem again and **identify**: a. THE GIVENS (the known quantities) b. WHAT NEEDS TO BE FOUND (the unknown quantities).
3.		Read the problem again & **separate the items** you wish to represent visually. Use a slash to help you.
4.	You DRAW the bar MODEL to help you see the problem and solve it.	**Represent one chunk** by at least **one rectangle or bar**. Label it with the known value or mark it as an unknown (use a"?" or variable").
5.		Return to the text to check the mathematical **relationships** between the bars of the model.
6.		**Adjust the size of the bars** to illustrate the relationships*. Does one or a group of bars find in others? How do they compare with each other? *You may go back and re-read the problem.
7	You TRANSLATE THE VISUAL into equations & solve them	**Translate** the relationships between **the bars** into mathematical sentences (**equations**).
8		**Compute the equations** and add the found information to the bar model.
9.		Repeat steps 5-7 until you solved the problem and have the **solution**.
10		*Does the solution make sense? Is it a reasonable number in the problem? In real life?*
11.		**State the solution** clearly, using a complete sentence that answers the question of the problem.

Resource created by Mirela CIOBANU, TDSB, 2014

Figure 1: *The Model Drawing Approach (can be downloaded from* ***mirelaciobanu.edublogs.org/teacher-resources/****)*

Let's consider the following problem and exemplify the use of Model Drawing. (*EQAO Junior Division Assessment Booklet, 2009–2010*, **www.eqao.com**)

> *The rates for Internet use offered by three companies are shown below.*
>
> - *Company A: $6.00 for every 90 minutes of use*
> - *Company B: $2.75 for every 45 minutes of use*
> - *Company C: $3.00 for every 60 minutes of use*
>
> *Which company offers the lowest rate per minute?*

Phase 1: Understand the problem.

In this case, the structure of this problem is based on the comparison between its elements. Students read the problem and identify the known and the unknown quantities. They use chunking to indicate the items they wish to represent visually.

Phase 2: Create the visual model and work with it to solve the problem.

A rectangular bar, labelled with the known value, represents each chunk. If that is unknown, the student can label it with a "?" or a letter (variable) (Figure 2).

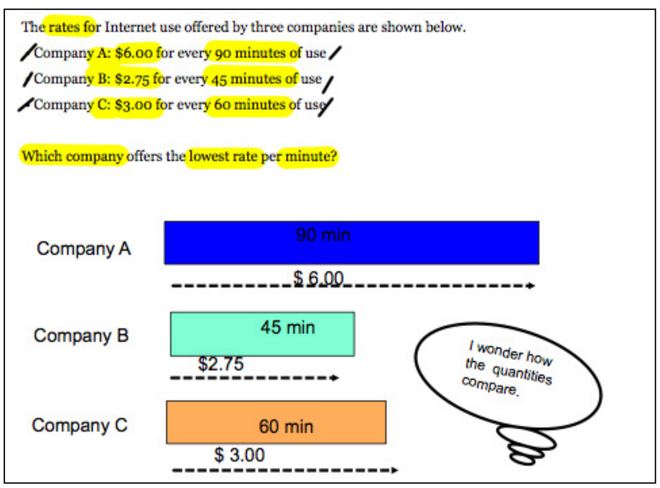


Figure 2: *Understanding the text of the problem and drawing the visual model*

The bars of the model are adjusted to represent the relationships between the information pieces in the problem. At this point, students pay attention to the numerical values and identify the part-whole comparison that is present in the structure of the problem and that allows them to convert the information into symbols, in math sentences. Looking at the diagram and numerical values, the students can ask themselves, *"Which value is the whole? Which one(s) can be found as part of another? How do the parts and the whole compare with each other?"* (Figure 3).

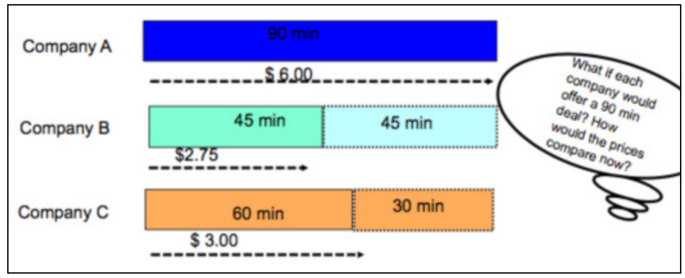


Figure 3: *The rectangular bars are adjusted and the relationships between them are explored*

Phase 3: Translate the visual into mathematical sentences involving symbols.

Students translate the visual relationships between the bars into mathematical sentences or equations, where applicable. As students discover new information, they add it to the model and continue to explore the relationships among its elements. Thus, students continue to use the model and convert in into symbolic mathematical statements until the problem is solved (Figure 4).

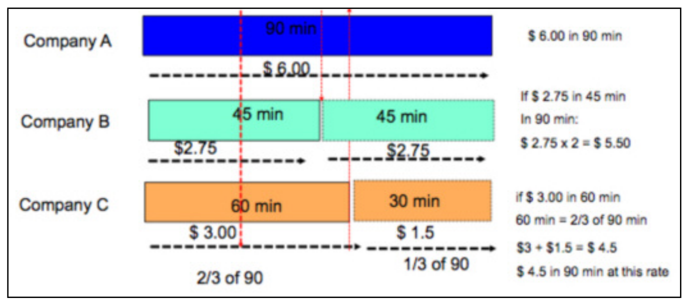


Figure 4: *The visual relationships between bars are translated further into mathematical sentences to the right as the model continues to be explored and new proportional relationships are discovered*

Finally, the students state the solution clearly, using a complete sentence (using either the discursive or symbolic registers).

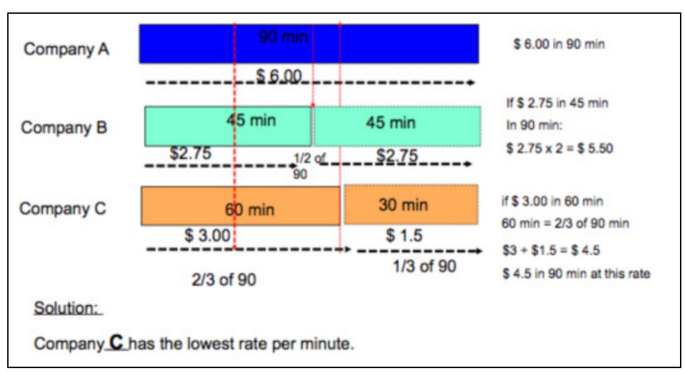


Figure 5: *The solution is stated clearly at the end in sentence form*

The Model Drawing approach is not the only type of visual representation that I support. The use of number line segments can be as effective. What I am suggesting is to give students an *effective* visual tool that they can use to both represent the problem visually in order to fully understand it and use it in order to solve it. Additionally, this approach enables students to organize their communication clearly and concisely and recognize the effectiveness of using mathematical statements, rather than using natural language and retelling word for word what calculations they employed, without mentioning what purpose those calculations served in solving the problem.

Resources for Classroom Teachers

Thinking Blocks: One resource I use a lot is Thinking Blocks. This is an interactive site that allows students to model and solve word problems, using the Model Drawing approach. The problem can be copied into the template and solved, or students can practise modelling problems provided by the site. Correct use of the Model Drawing approach needs modelling and it takes a lot of practice. (**www.mathplayground.com/ThinkingBlocks/thinking_blocks_modeling%20_tool.html**)

The Singapore Maths Teacher is a site that provides free lessons on how to use model drawing. The word problems are levelled in order of difficulty (three levels plus an enrichment one) for each year from Primary 3 to Primary 6 (Singapore curriculum or year 3 to year 7, USA). (**www.thesingaporemaths.com/**)

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▲ Mb4T (Mathematics by and for Teachers): The Importance of Connecting Patterns to Multiple Algebraic Descriptions

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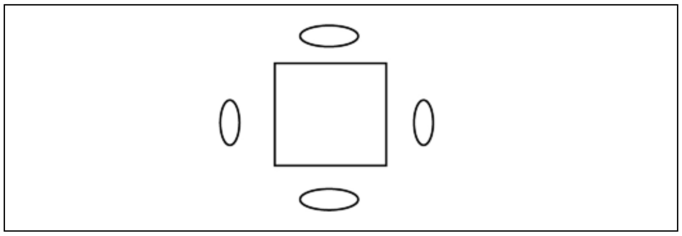
As discussed in the previous column, the Mathematics curriculum (Ontario Ministry of Education, 2005) and researchers such as Boaler (2014) are encouraging the support of students' *multiple* ways of seeing and describing patterns. However, those of us originally schooled in a traditional paradigm may have to continue to work hard to avoid our pre-programmed belief in the "one right answer" approach. In fact, it's important to hold tight to the idea that the importance of connecting visual patterns to algebraic expressions isn't really just about finding the expressions themselves. Rather, just as we know flexible understandings of early number provide a solid foundation for understanding operations, so does a flexible understanding of the creation of pattern rules provide a solid foundation for the development of algebraic thinking.

As discussed in detail in the previous column, the emphasis on creating and connecting different rules based on different ways of visualizing patterns moves problem solving, reasoning, communicating, connecting, and reflecting to the foreground of our classroom discourse. It's important to resist a focus on one particular way of setting up an algebraic rule, such as the "multiplier plus constant" idea. While such algebraic simplifications may at times be useful later due to their possible computational efficiency (and hence tempting to teachers who may find them "simpler"), forcing elementary children to use a particular style of pattern rule may not support our goal of developing flexible algebraic thinking. In the current column, we'll continue to explore this idea of the benefits of focusing on multiple methods, using an example drawn from the new mathematics resource for teachers, Kajander and Boland (2014).

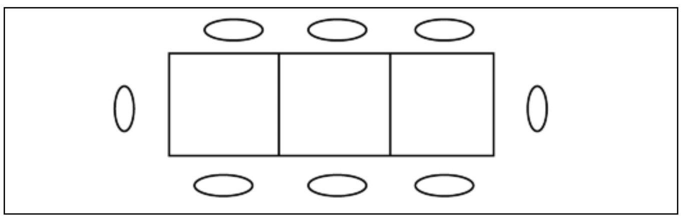
The Banquet Tables Example

Let's explore a possible classroom example.

Your school has square tables available for events. Initially, 4 chairs fit around a table:



For the year-end dinner, tables will be lined up in a long row in the gym. So for 2 tables, there will be a total of 6 chairs. Here is the chair arrangement for 3 tables:



Let's say we are interested in finding out how many people we can seat for a given number of tables lined up in this way. One way to keep track of this information is to use a table that compares tables to chairs (such as an Input/Output chart, which typically translates to an *x*/*y* table at the secondary level). However, we need to be aware that it may in fact be the *geometric* illustration that provides the most richness for the generation of multiple pattern rules, and hence for now, the geometric model should be retained.

To begin working on the problem, we might start by encouraging children to sketch or model the next few arrangements of tables. An early observation might be that "for every new table, we can add two more chairs." This observation about the pattern growth is an important one to note and discuss—such recursive thinking exactly aligns with the concept of the rate of change or slope, a fundamental of early secondary-level linear relations.

The goal of the problem might be to have students understand how to find a way to directly connect the number of tables (which may be referred to as the input value, picture number, or frame number, and later, the independent variable) to the number of chairs (the output value, which is eventually called the dependent variable). It may be helpful to explore a further series of pictures here.

We might begin to explore the problem with students by posing an initial problem, namely by asking them to find different ways to calculate how many chairs fit around 10 tables arranged in the manner shown in the diagrams. Before reading further, find as many ways as possible yourself to calculate the number of chairs, by working to visualize the arrangement and grouping of chairs differently each time (but in a way that will work in a similar manner for other numbers of tables).

Solutions to the 10-Tables Problem

You may be surprised at what you find when you share your thinking and solutions with a partner. The richness of this problem is in the *various* possible solutions, and how they relate to each other. The existence of these multiple solutions is why classroom discussion and sharing after an activity is such a crucial aspect of conceptual learning. Teachers who regularly engage in such activities report that sometimes the traditionally "less successful" students come up with ingenious ways to construct rules; however, in my own classroom observations, I have at times seen such non-standard approaches "corrected" to an algebraic equivalent that the teacher finds "simpler." Rather, non-standard approaches should be shared and celebrated; they are the very examples which allow us to encourage flexible thinking among a class group. Indeed, there is some evidence that "correcting" a student by changing his or her thinking method away from one fruitful method to another, for no particular reason, is damaging.

If possible, share your ideas for rules to count the chairs for the previous 10-tables example with a colleague. If you do not have colleagues handy with whom to share, consider the following possible student solutions for the problem. Work to follow the reasoning in each case, and compare it with your own reasoning.

Examples of Possible Student Reasoning

The following represent different avenues of thinking that might emerge from different students. The list is not exhaustive!

1. I knew from the smaller number of tables that 2 chairs are added for every new table. I already knew from the diagrams that 3 tables had 8 chairs, so I added

2 chairs for each of the 7 new tables after 3 (to get 10 tables). The number of chairs for 10 tables is $8 + 2 + 2 + 2 + 2 + 2 + 2 + 2$, or 22.

2. I looked for a relationship between the number of tables and the number of chairs. Leaving off the chairs on each end for now, the number of chairs is 2 times the number of tables. I then added 2 for the chairs on each end. So the answer is $10 \times 2 + 2 = 22$.
3. I pretended that each of the 10 tables had 4 chairs. That's $10 \times 4 = 40$. But there aren't any chairs between the tables (where they join). So for every side where they join, we lose 1 chair from each side, so that's 2 chairs. For 10 tables, there are 9 joins (I almost thought it was 10, but when I drew it I saw it was 9). I subtracted 9 joins × 2 chairs, or 18 chairs. The answer is $40 - 18 = 22$.
4. One table has 4 chairs. When I add a new table, I add 3 chairs, but I lose the one where the tables touch. For 2 tables, it is $4 + 3 - 1$. For 3 tables, it is $4 + 3 - 1 + 3 - 1$. For 4 tables, it is $4 + 3 - 1 + 3 - 1 + 3 - 1$. I do the "$+ 3 - 1$" calculation for each of the 9 new tables. The answer is $4 + 9 \times (3 - 1) = 4 + 9 \times 2 = 4 + 18 = 22$.

Discussion

The methods just described that are similar to your own will be easier for you to understand than those that use different reasoning. Similarly, others may have to work to understand *your* thinking! The art of being an effective teacher is developing the skill of understanding *others'* solution methods, rather than just the one that you prefer. A method that is easy for one person to use and understand is not always the easiest for another. It is challenging to move away from our own preferred solution method and embrace another, and to realize that another person's chosen method might be the easiest way for *him* or *her* to think about the problem.

Practise using a solution method provided that was not your own. Choose one of the solution methods provided in solutions 1. to 4. or one from a colleague that you did not think of yourself, and apply this reasoning to the same 10-tables task. Of course, you should get the same answer as you did using your original method! Compare the thinking and reasoning involved.

So far, we have looked at the relationship of chairs to exactly 10 tables. However, all of the geometric reasoning in the previous methods can be *generalized* to other numbers of tables. For example, we might practise by choosing one of the methods 1. to 4. previously and applying it to a different case—say 8 tables or 13 tables. Eventually students can recognize that there is a consistent relationship between the number of chairs and the number of tables. This insight is the key idea of the task. Indeed, we see the initial understanding of the idea that even though the number of tables may be unknown, the relationship of tables to chairs stays the same. This is a critical realization.

Generalizing Our Thinking

Once students realize that the *relationship* (however they constructed it) of tables to chairs stays the same, the move to a generalized rule is relatively seamless. We can use a symbol such as n to represent an unknown number of tables. You might practise by *generalizing* a few of the student counting methods as provided earlier yourself. (The possibilities are provided at the end of the article for checking.) Note that the first method begins with 7 tables, which may seem a little arbitrary when creating a generalization. Methods 2. to 4. may be easier to generalize and you may want to start with these.

One aspect of the problem was that there was uncertainty about the number of tables, and, hence, about the number of chairs; as a result, we had to examine the problem for a *varying* number of tables and chairs. For problems in which there is such uncertainty, we often refer to the symbols representing the unknown amounts as *variables* because there are many possibilities for their values; in other words, they can *vary* in value. I have found that this change from a *single* case such as 10 tables to the use of a letter such as x to represent *many* different possible values can be a conceptual roadblock for some students at the secondary level; thus, these early pattern rule construction experiences are a key developmental piece for elementary students.

The conceptual difficulty for some students is often the connection between the idea of 3 tables, 4 tables, 5 tables, and then moving to *any* number of tables, where the *any* number is represented by *x* or *n*. If students are not ready to see that x can take on a number of values, they need to work through more examples, and examine more diagrams as the number of tables increases.

For a classroom video of students enacting a lesson with the intent similar to the ideas described, see the first lesson in Boaler and Humphreys (2005), also available on YouTube.

To conclude, here are some possible generalized solutions to the example, numbered as they align with the earlier 10-tables case.

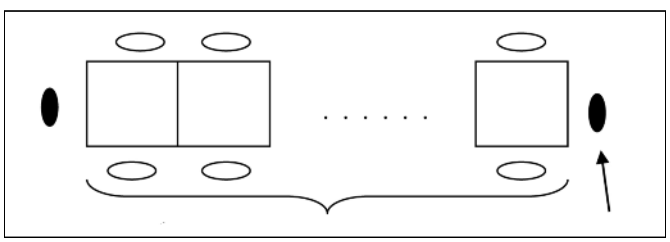
Possible Solutions to the General Pattern Rule

Note that the *order* in which a teacher chooses students to share their thinking during the consolidation phase of a lesson is important; a relatively simple solution such as the one in solution 2. might be a good starting point. However, solution 1., as follows, may be harder to generalize and therefore might be left until later in the sharing. These choices will depend on students' work on any given day, and flexibility is key.

1. Since the 3-table case was the starting point for this rule, we start with the chairs needed for 3 tables, which is 8. Then what is being added is the number of chairs for the number of new tables—which is the total number of tables *less 3*. So the rule could be written as 8 + 2 + 2 + ... + 2 the 2 is added (number of tables – 3) times.

   After some discussion, students might be willing to accept that if we want a rule for n tables, we have to add chairs (n – 3) times. So the number of new chairs is 2 × (n – 3). The rule becomes 8 + (n – 3) × 2.

2. We notice that there is a chair on 2 sides of each table (so the number of these chairs is 2 times the number of tables), and 1 chair on each of the 2 ends of the row of tables (added just once). So the 10-table case is 2 × 10 + 2. The following model illustrates the idea in general:



   twice the number of chairs as tables plus 2 extra chairs (shown in black on each end).

   Hence, the rule for counting chairs for n tables becomes 2 × n + 2.

3. Here, the method for 10 tables was 10 × 4 – (9 × 2). A key idea is that the "9" is "1 less than the number of tables." (Remember that the thinking is "4 chairs for each table, less 2 chairs at each join. The joins are 1 less than the number of tables.")

   So the rule becomes: n × 4 – ((n – 1) × 2).

   Don't forget to hold tight to the ideas described at the beginning of the article, by resisting that urge to "correct" this rule to something that might be seen by the teacher as "simpler"!

4. Lastly, the rule 4 + (3 – 1) + (3 – 1) + ... + (3 – 1), which students may or may not be willing to simplify to: 4 + 2 + 2 + 2 ... + 2.

   It must be understood that the (+2) is added for all but the first table. "All but the first table" might be represented as n – 1, if there are n tables.

   The rule can then become: 4 + (n – 1) × 2.

   Alternately, it could remain as: 4 + (n – 1) × (3 – 1), or even, 4 + (3 – 1) × (n – 1).

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▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Timothy Sibbald, Nipissing University

Projects: Tim is currently working on a project that is examining the statistical structure of Ontario classrooms found in the EQAO data sets. According to Dr. Sibbald, a common misconception is that classroom achievement is bell-shaped, whereas this actually occurs relatively rarely. A much more common occurrence, he further notes, is the bimodal or multi-modal classroom. Whether this is observed through time, or at different grade levels, is currently being investigated. ▲

▲ Provincial Digital Learning Resources – What's New? GSP Files – Gems from the Past!




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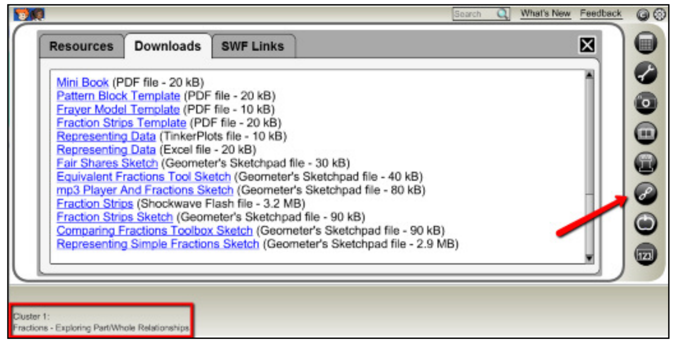
Greg is from the Simcoe Muskoka Catholic District School Board and Ross is from the Near North District School Board. Both are currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS project, in conjunction with their colleague, Agnes Grafton, from the Brant Haldimand Norfolk Catholic District School Board. Markus Wolski is a teacher with the Bluewater District School Board and has been working on the CLIPS project for the past two years.

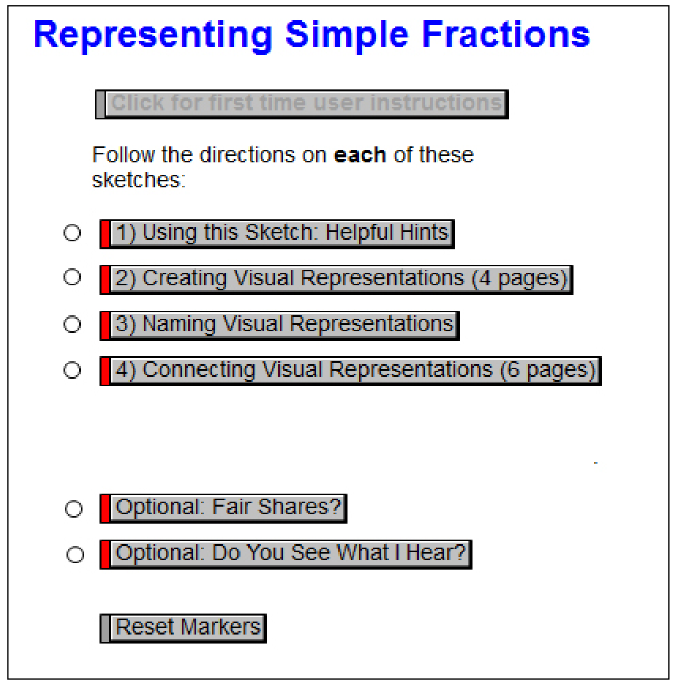
In the December 2014 issue, we announced that the Geometer's Sketchpad® (GSP) files at MathCLIPS.ca have all been updated to version 5 and optimized for the Sketchpad Explorer iPad App. We also outlined how to use a Puffin App to download sketches to an iPad. In the process of testing these sketches, we were reminded about how great they are. Many of them were created as part of the PRISM project (see **oame.on.ca/PRISMNEO/**) with the help of Sketchpad gurus Nicholas Jackiw, Nathalie Sinclair, Scott Steketee, and Audrey Weeks. In this issue, we feature three of these sketches.

1. Representing Simple Fraction Sketch

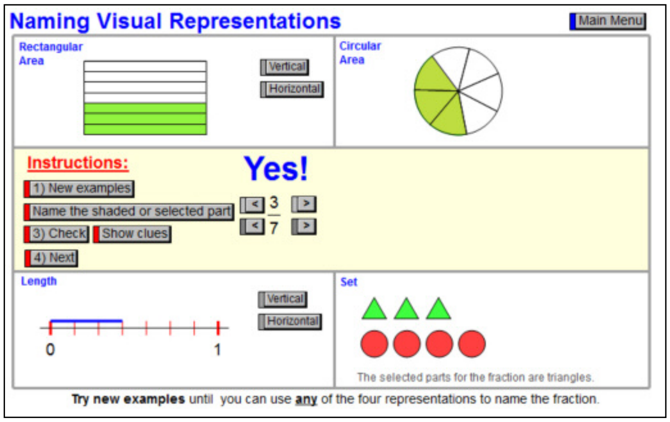
This sketch is one of the resources developed as an additional support for the Fractions – Exploring Part/Whole Relationships cluster. To find the sketch, click on the resources button from within this cluster. (Note the list of other useful fraction resources located here.)



Alternatively, the sketch can be located by clicking on the magnifying glass and searching for it. When it opens, you are presented with a menu of pages.

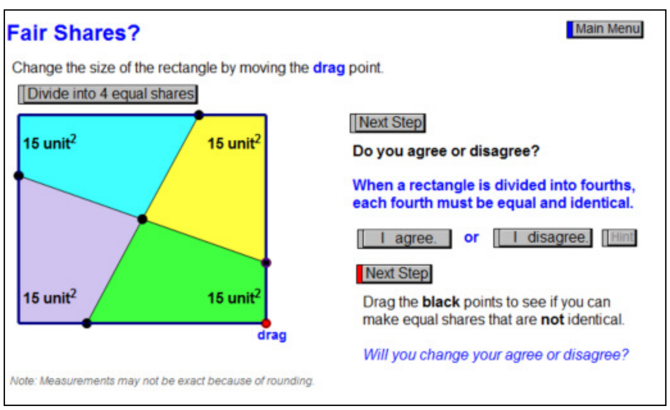


This sketch provides students with the opportunity to both create and name various simple fractions, using rectangular and circular area models, in addition to number lines and set models. The fraction values range from 0 to 1 and include unit fractions, proper fractions (some in unreduced form), as well as fractions equivalent to 1.

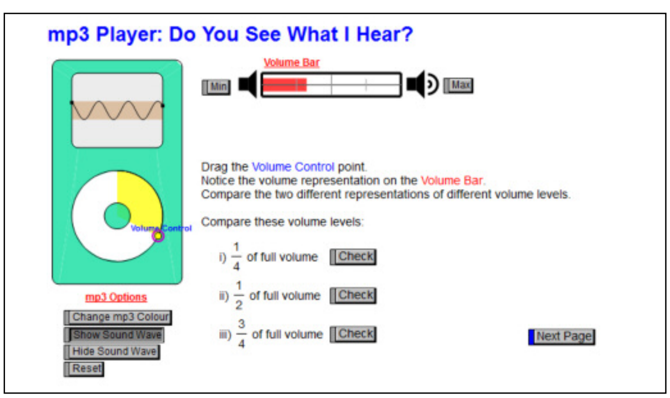


The power of this sketch lies in the opportunity to make connections between various representations as the students explore what happens to each representation as changes to the numerator and denominator are made. Students are encouraged to examine the special cases when the numerator is equal to 0, equal to the denominator, and equal to half of the denominator.

This sketch also includes two optional tools. The Fair Shares sketch allows students to explore the idea that equal areas need not necessarily be congruent, an idea that many may find challenging.

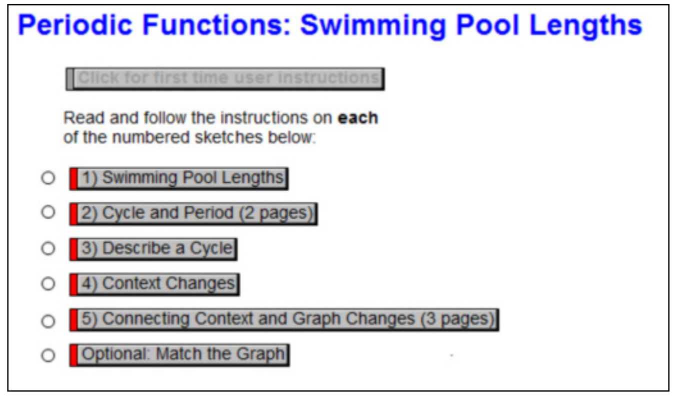


The *mp3 Player* sketch helps students make connections between the various representations as the volume is adjusted.



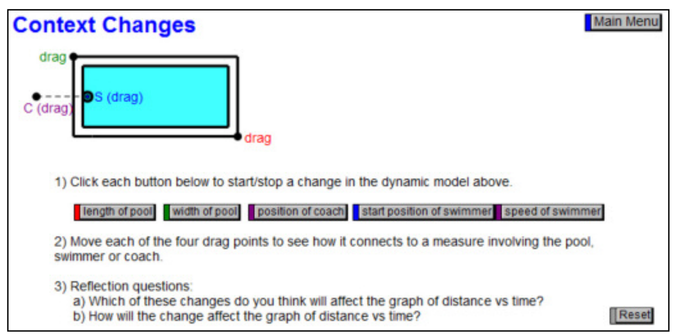
2. Swimming Pool Lengths Sketch

This sketch is activity 1.3 of the Trigonometric Functions – Graphing Sine (degrees) cluster (**www.mathclips.ca/index.html?cluster=2&clip=1&activity=3**).



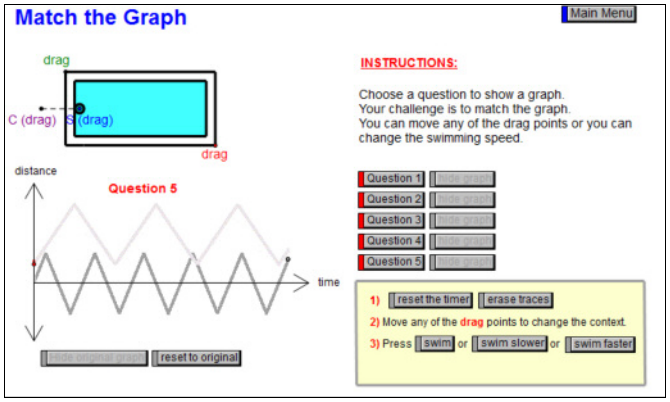
It is one of the activities about “Graphing, Identifying, Describing Periodic Functions.” It opens with a menu of pages. The first step is to open the Swimming Pool Lengths page, where we are introduced to the context of a swimmer doing lengths under the watchful eye of a coach. In steps 2 and 3, the student applies the key terms, *cycle* and *period*, to the context.

The heart of this sketch are the pages related to changing the context. First, students can use buttons or drag points to affect the physical set-up. They are asked to think about how such changes will affect the graph, if at all.



In step 5 of the menu, students then can make changes to the context and observe how the graph changes. A teacher might want to provide some sort of recording sheet to accompany the sketch so that the students can organize their investigations and reflections, and so that mathematically precise language can be used to describe the resulting graphs as compared to the original one.

Although indicated as optional, the “Match the Graph” page will provoke a lot of thinking. Students are given a question, which is a transformed graph, and must adapt the context to match it.



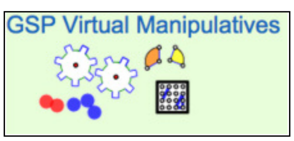
In question 5, students will have to realize that the swimmer must swim faster to have a shorter period. Positioning the swimmer and the coach requires some important insight into the relationship between the context and the graphical representation.

The consolidation activity 1.5 for this Clip is a quiz with examples and non-examples of periodic functions, which has proven to be quite effective at giving students an intuitive notion of what a periodic function is.

Later, in CLIP 5, students have other opportunities to investigate how changes in context affect the graph of sinusoidal functions, and use standard transformation terminology to describe those changes (in particular, see activity 5.3).

Not only is this sketch powerful, but the entire instructional sequence used to introduce periodic functions, graph the sine function, and apply it in context is compelling.

3. The UberSketch Collection of GSP Virtual Manipulatives

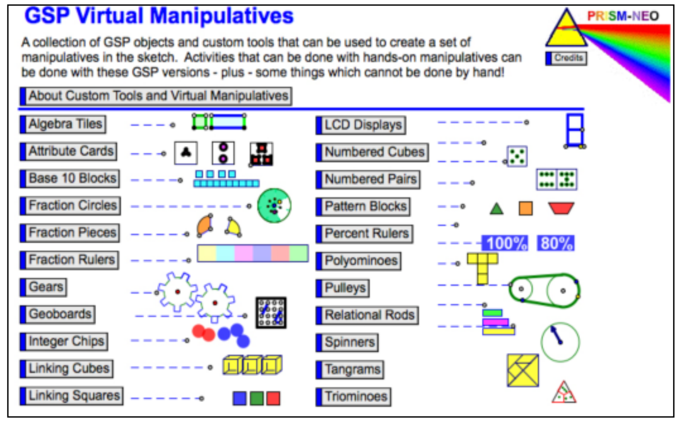


The uberSketch is an extensive collection of virtual manipulatives and Sketchpad Tools curated by Greg Clarke and featured on wikipedia.

Access the uberSketch by going to mathies.ca, clicking on Learning Tools, and finding the button for the GSP Virtual Manipulatives.

The sketch opens up with a menu listing various virtual manipulatives that can be created with GSP. The custom tools to build each of the tools are included in the sketch. When you visit a specific virtual manipulative page, there is an introductory page describing its purpose and typical usage, and then two more in-depth pages on how to use the custom tools associated with it.

Take some time to explore this sketch, and think about how you might use some of these virtual manipulatives.



Further Exploration

To explore other sketches on your own, you could do the following:

1. Visit mathclips.ca and search for "gsp."
2. Visit **oame.on.ca/PRISMNEO/** for sketches designed for Grades 7–9.
3. Visit **oame.on.ca/PRISMNEO/Grade11/** for sketches designed for Grade 11, including sketches related to Exponential Functions, Simple and Compound Interest, Trigonometry, Quadratics, and Inverse functions. The fnUberSketch is another compilation of powerful Sketchpad tools related to functions and graphing.
4. Visit the Sine of the Times blog (**blog.keycurriculum.com**), the Engaging Math blog (**engaging-math.blogspot.ca/search/label/GSP**), the Mathfest blog (**mathfest.blogspot.ca/search/label/sketchpad**), or the Sketchpad for Younger Learners page (**www.dynamicgeometry.com/General_Resources/Classroom_Activities/KCPT/Activities_for_Young_Learners/Sketchpad_for_Grades_3-5/Activities.html**) for posts about GSP, with sketches available for download.

These sketches may not be recently created, but perhaps they are underappreciated enough to warrant being the subject of a "What's New" column, and as a result, will become hugely appreciated by readers and their students!

The *Rekenrek* app discussed in our March 2014 column is now available on the App Store, Google Play Store, and for desktop computers (**www.mathies.ca/learningTools.php**). ▲

▲ Fields Institute MathEd Forum Report

STEWART CRAVEN
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Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

By the time you read this, we will have had our annual research day. This day is devoted to highlighting some of the many initiatives involving the study of mathematics education across the province and beyond. The following are full descriptions of the first and the last presentation, along with the titles of the other three topics:

1. Zekeriya Karadag, Seyda Birni, Dragana Martinovic, Ozlem Deniz (Bayburt University, Turkey, and University of Windsor, Canada)

   Abstract: *This talk reports on the first step of a multi-step international research project. On the one hand, the scholars suggest three main features provided by the new generation of learning softwares: visual, dynamic, and explorative. On the other hand, the literature argues that the school-age generation is the generation [of] so-called Digital Natives, growing up with technology, and their way of learning [has] been evolving parallel to the mass exposure of technology. The claim is that Digital Natives are becoming visual, dynamic, and explorative learners. Therefore, we decided to look at the intersection of these two claims, and [aim to understand] how DIMLE meet the evolving expectations of new generations. The research is intended to explore the ways in which [the] new generation makes use of mathematics learning environments. The core research question is set: Do they really learn mathematics in a visual, dynamic, and explorative manner?*

2. Yasmine Abtahi (University of Ottawa): "A quarter wouldn't be that": Mathematical tools and the emergence of ZPD
3. Chester Weatherby, Douglas Woolford, Donna Kotsopoulos (Wilfrid Laurier University): Streaming protocols for university-level mathematics: The effects of placement tests on first-year calculus achievement
4. Kevin Thomas, Ami Mamolo (University of Ontario Institute of Technology): Thinking about teaching math for social justice
5. Ann Kajander (Lakehead University): Tears, trials, and transformations: The requirement of deep teacher knowledge development in mathematics education

   Abstract: *Appropriate mathematical knowledge as needed for elementary classroom teaching has been an important research topic during the last decade. Data from the most recent two years of a longitudinal database related to prospective teacher mathematical understanding will be presented. Support for recommendations for program changes in teacher education programs will be provided via quantitative and qualitative data, including the participants' own voices.*

I would like to challenge you to consider coming to our meetings (agendae and information can be found at **www.fields.utoronto.ca/programs/mathed/forum/**) I would also like to invite you to make suggestions about what themes you think should be addressed by the Fields MathEd Forum (drop me an email at **numerate citizen@mac.com**). ▲

▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Trevor Brown, Tyndale University College

Projects: Trevor is working with some teachers at a school for orphans in the southern part of India to see if he can give the teachers and students the needed skills so that the students can attend some British universities. Trevor has spent two summers with these teachers and students, and will be returning to India in June 2015. He is also working with a group of teachers who have just received some funding to develop rich resources on proportional reasoning for Grades 7 and 8. ▲

▲ The E-Brock Bugs Computer Game: What If Becoming a (Better) Mathematician Were a Fun-Filled Adventure?

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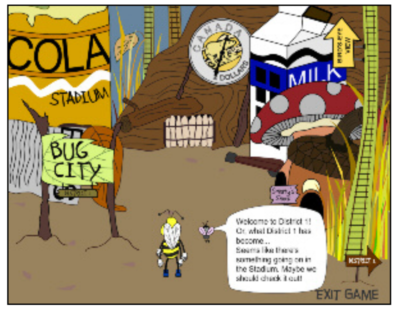
Laura Broley is a Master's student at Université de Montréal, where she is studying mathematics and researching the use of computer technology in mathematics education. She created E-Brock Bugs during the completion of her B.Sc. Honours in Mathematics Integrated with Computers and Applications at Brock University. Chantal Buteau is Associate Professor of mathematics at Brock University, where she has taught for over ten years a course in which students learn to design, program, and use interactive environments to investigate mathematical concepts, conjectures, theorems, or real-world applications. Her research interests include the use of digital technologies in mathematics learning. Eric Muller is Professor Emeritus of mathematics at Brock University. During his many years at Brock, he was actively involved with OAME and was a founding member of the Golden Section. He was honoured to receive the OAME Life Membership Award. His sincere hopes are that E-Brock Bugs will be as well received in Ontario mathematics classrooms as was the original Brock Bugs board game.

Abstract

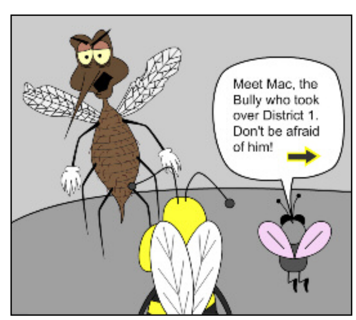
October 2013 marked the launch of E-Brock Bugs©, a free online mathematics computer game based on a board game developed by Muller in the 1980s (Muller, 1987). Like its predecessor, E-Brock Bugs may serve a wide audience: for instance, it could be used to engage younger students in learning probability concepts through experimentation, or provide theoretical support for high school learners in the MDM4U course. In this paper, we present the E-Brock Bugs adventure, followed by a breakdown of the mathematics involved, a discussion of the guiding educational philosophy, and some additional information about the game. Ultimately, through the implementation of Keith Devlin's (2011) principles of "good" math video-game design, we hope that the adventure is not just about learning how to do math in a mechanical manipulating symbols sense; it is about learning how to become a (better) mathematician.

E-Brock Bugs, the Adventure

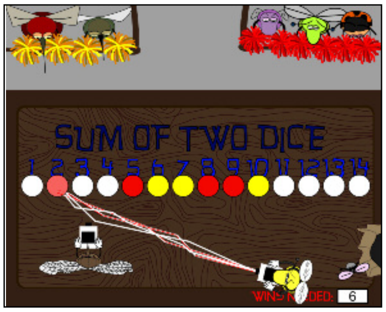
Dramatic music plays in the background as I read: Bug City had always been a peaceful place to live... until... one day... the Bullies came to town. The simple bugs of the city are now dominated by the Bullies' intelligence; they need my help! So, I pick my character (the bee, of course!), name it (what better than "Bumble"), and start out on my journey to save Bug City! A strange little bug named Bugzy immediately greets me and takes me to District 1, which appears dark, dusty, and deserted; the sad look of the place makes me feel a little sad myself. But, as Bugzy says, there's no time to lose! I'm ready to face these Bullies! But, where are they?



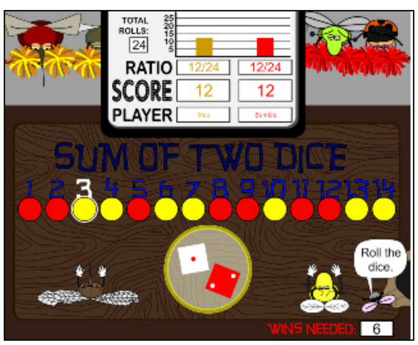
I can hear some cheering coming from Cola Stadium, so I decide to check it out. Sure enough, that's where I find him: Mac, the Mosquito, the Bully who took over District 1. Even if he looks a little frightening, he can't scare me! Bring it on! I'm ready! After entering the stadium bravely, I find myself on a large game field surrounded by a mixed crowd. The nerves start to set in. But, looking at all the bugs that are depending on me, I am urged forward to take my place across from my opponent, Mac. I am given a strange device and told that it has some sort of a laser in it... but, I don't even know how this game works! I learn very quickly how lucky I am to have Bugzy on my side. He explains: Mac and I will take turns using our lasers to light up numbers from one to fourteen until all of them have been chosen. Then, we will roll two dice, add the numbers on their faces, and whoever chose the number corresponding to the sum will get a point. After 25 rolls, the bug with the most points wins.



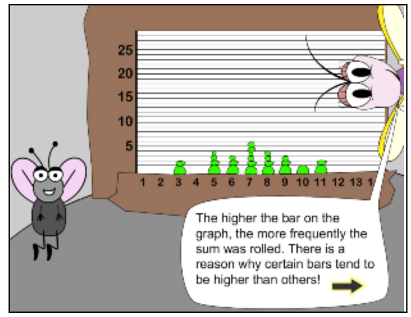
When we start playing, I'm not really sure which number to choose. Wait a minute, why is there a one on the board? Two regular dice can never add to one, can they? I'm definitely not choosing one. Oh, and not thirteen or fourteen either! Let's see... You can get a sum of eight by rolling two fours, a five and a three, a two and a six... It seems like that would be a good choice! Let's go for it! Mac immediately chooses seven next, which I follow with a nine. His next choice of six makes me think five could be a good third choice. And, on my fourth choice, I decide to go with my lucky number: two. Unfortunately, I eventually have to choose one. But, at least Mac got both thirteen and fourteen! Ha ha! Rolling the dice comes next. At first, I look carefully at each roll to make sure the game isn't rigged! I notice, however, that I can roll the dice very quickly, which makes the stadium fill with cheers from mine or Mac's cheerleaders. I have to slow down near the end to calm my nerves. 12-12, tie game: it's so close! But, in the end, it's me who wins! YES! Only five more and I will have already saved District 1!



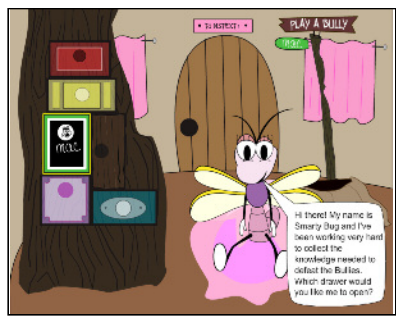
After the game, Bugzy pulls me aside and, once again, I realize how helpful this little guy is! He had kept track of all the sums that were rolled during the game. I notice that some of the numbers around the middle of the board were rolled a lot, while my lucky number two wasn't rolled at all. An unfamiliar bug pops out of nowhere and tells me that there is a reason why certain sums were rolled more often than others. She doesn't stay long, though; who is she and where did she go? I regain my focus: let's go beat Mac again! This time, based on Bugzy's counts, I choose a different set of numbers. Unfortunately, I'm surprised to see that it doesn't work out well for me. Mac wins, 15-10, and he isn't very nice about it either. I guess there's still five more wins to go....



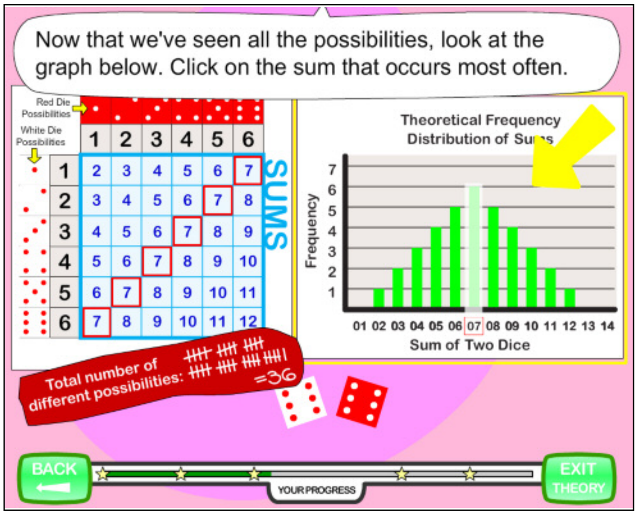
Just then, as I'm feeling a little discouraged, Bugzy whispers to me that there is some secret tunnel that we could use to get to Smarty, a bug that may be able to give us some help. I say: Awesome! We can use all the help we can get! The tunnel is a little dark and dingy, but what appears on the other side is anything but: it's bright, colourful, and full of cheerful music. And there's that bug I saw earlier! You must be Smarty! You can help me? Alright! I definitely want to hear about the theory behind Mac's game.



Smarty immediately goes to her tree dresser and pulls out a whole bunch of materials from a drawer marked with Mac's face; this girl is organized! One of the many items I see in her pile is a miniature model of Cola Stadium's playing field and some chips that I use to show her my current game strategy. This is so cool! Not only is this just a smaller version of Mac's game, but my chips have my own face on them! Soon after, she puts the model aside and brings out some white leaf sheets. Together, we start filling in a chart and graph at the same time: I roll the dice and Smarty records the results. I watch as the patterns develop in the chart and the bars rise on the graph. Once we've found all the possible rolls, I can finally see what's going on. There are actually six different ways I can roll a sum of seven, which makes it the best spot on the game field. It's no wonder that my favourite number two wasn't rolled: it only has one option! And, like I predicted, there is no way to roll a sum of one, thirteen, or fourteen. I even explain all of this to Smarty, using the charts we constructed.



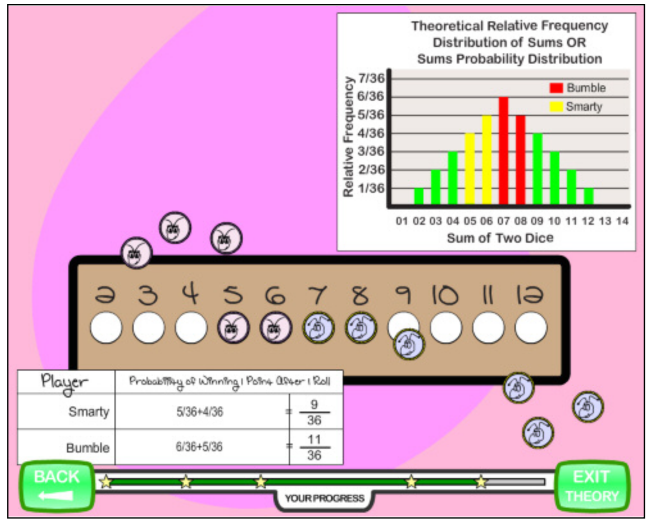
Before we can move on, Bugzy decides to speak up. He recognizes a similarity between the graph we made and the counts from the last game I played. I can see some similarities too, Bugzy! The difference, Smarty explains, is that Bugzy's counts relate to an actual game, while the graph we just created gives the theoretical distribution of the sums. Huh? Oh I get it! With Smarty, we're talking about what should happen in theory, which



isn't always the same as what happens in real life. Great work, Bugzy!

Smarty lays down another graph beside the one we already have... But, wait... Now it's my turn to spot the differences! Look at the titles: one graph says "frequency distribution," while the other says "probability distribution." What do you mean by probability, Smarty? Oh, okay! Probability is just a fancy word for talking about the likelihood that a sum will occur. The higher the bar on the graph, the higher the probability... The higher the probability, the more likely the sum... And the more likely the sum, the better it is for me to light it up with my laser! Using this idea, I practise against Smarty and learn the best strategy to win against Mac. I really feel like I've got this now!

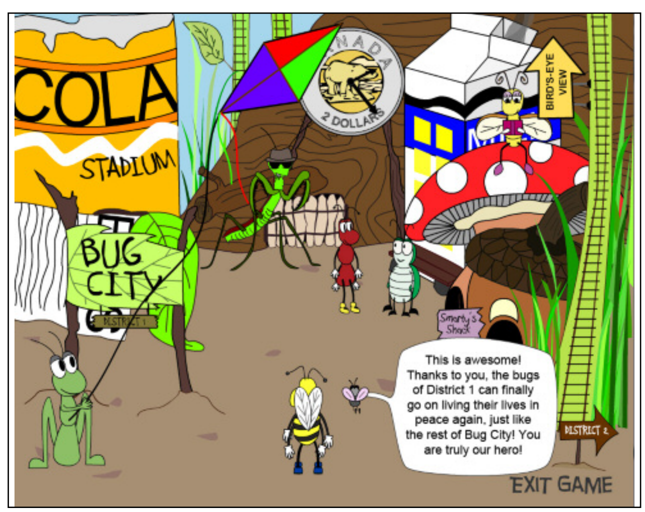
When I tell Smarty that I think I'm ready, she mentions that she has created some exercise leaves that I can use to practise what we just discussed. I even have two options: a small leaf, which contains an "easy" set of exercises, and a large leaf, which contains the more



"difficult" exercises. I breeze through the small leaf because I am able to use the charts we just finished creating together; the larger leaf, on the other hand, is a little more difficult. Luckily, the leaf sheets are randomized and I am able to practise a new set of questions as many times as I want!

After visiting Smarty, I feel ready to save not only District 1, but all of Bug City. Mac, Bash, Fitz, Trickz, Crazee, and finally, Wicked, are no match for me! With the constant help of Bugzy and Smarty, I am able to make it to the very end, where I finally see the life of freedom restored to the simple bugs in each district—bugs roaming about, the sun in the sky, joyful music filling the streets. It's a happy place... But, there's still something off about it. That mysterious bug lurking in the back over there... Who is he?

All of a sudden, the music stops, and Bugzy and I are captured beneath some strange object. Hey, hey, what's going on here? What? I'm being captured? But, I defeated all of the Bullies? Who are you?... Dr. P...? You were the mastermind behind all of this? Wait! Where are



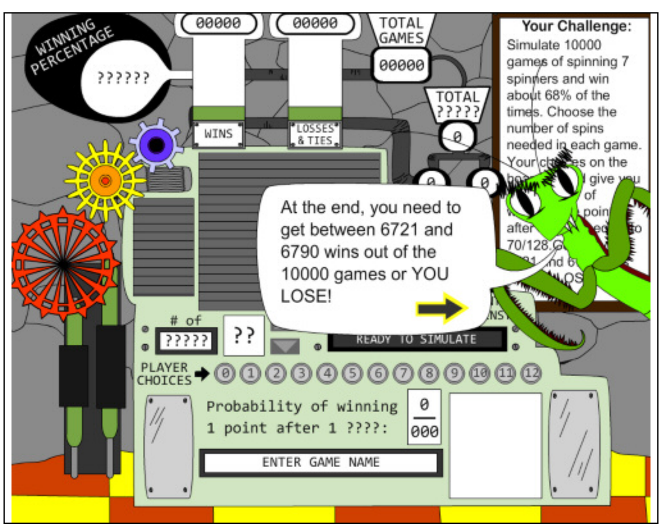
Increased Difficulty and Engagement

	DISTRICT	GAME	PROBABILITY CONCEPT(S)	MAIN IDEAS
FIXED DISTRIBUTIONS	1	Sum of Two Dice	probability distribution; addition rule	I. frequency and relative frequency bar graphs; II. $\frac{\text{frequency}}{\text{total outcomes}}$ (Districts 1–5)
	2	Sum of Two Fibonacci Dice	asymmetric probability distribution	
	3	Sum of Two Drawn Balls	independent/dependent events; product rule	
RANDOMIZED DISTRIBUTIONS	4	Sum of Two Spinners	equally/not equally likely events	
	5	Spinning Seven Spinners	Binomial distribution (n = 7)	
	6	Sum of Two Dice with Point Values	expected value	III. theoretical versus empirical probability (all districts)
	Finale	Simulation Challenge	Binomial distribution (n to be determined)	

you taking me? What is this place? A final challenge? What? A simulation machine? 10 000 games, and I need to win between 6721 and 6790 of them? What? Smarty! Where are you?....

The Mathematics in E-Brock Bugs

Through six different games and a final simulation challenge, players of E-Brock Bugs are exposed to the several basic probability concepts and three main ideas summarized in the following table (adapted from Broley, Buteau, & Muller, forthcoming; Broley, 2013). Note that the first two main ideas (i.e., the construction of frequency and relative frequency bar graphs and the calculation of probabilities by placing the frequency of an event over the total number of outcomes) are emphasized throughout the first five districts, while the third notion (i.e., the difference between a theoretical and an empirical probability) is highlighted throughout the entire game. In addition, the games and concepts have been carefully organized in such a way that their level of difficulty and engagement gradually increases. For instance, the final three games and simulation challenge each involve a randomized distribution (as opposed to a fixed distribution), which means that these cases generally require much more work (outside of the game) to determine the best possible strategy. For a more detailed description of the mathematics in the game, see the E-Brock Bugs teacher document (Broley, Buteau, & Muller, 2013b).



The Educational Philosophy of E-Brock Bugs

E-Brock Bugs was designed with the goal of classifying it among those educational games that are called "epistemic," "where players think and act like real-world professionals" (Bagley & Shaffer, 2009, p. 36). In an epistemic mathematics computer game, "[t]he player becomes a mathematician and problem solver within the context of the game" (Offenholley, 2011, p. 45). In other words, this kind of game goes beyond drilling basic facts and skills, and encourages players to develop their mathematical thinking and mathematical identity.

In his 2011 book, *Mathematics education for a new era: Video games as a medium for learning,* mathematician Keith Devlin outlines several principles of "good" video-game design that could be used to categorize a game as "epistemic." Combined with the constructivist approach (Piaget, 1967; Vygotsky, 1978), these principles serve as the main components of the educational philosophy behind E-Brock Bugs:

1. Naturally arising, meaningful, and sensible mathematics: “The mathematics should not be hidden; the players should know they are doing math! But that math should arise naturally in the game, it should have meaning in the game, and it should make sense in the game” (Devlin, 2011, p. 127).
2. Learning by doing: Players are “never put in a position of having to ‘learn something’ prior to playing the game in order to play the game” (Devlin, 2011, p. 128). Practice comes before theory so that players may develop their own understandings of their experiences.
3. Self-paced learning: The player is always able “to explore new concepts and practice new techniques at his or her own pace” (Devlin, 2011, p. 29).
4. Progressing through exploration: Players may “explore, try things out, and become familiar with new ideas and skills” (Devlin, 2011, p. 129). This way, they may have more fun and develop more powerful knowledge. As the old saying goes, “You tell me, I forget; you show me, I remember; you let me discover, and I know” (Devlin, 2011, p. 99).
5. Learning new skills and facts for immediate use: The player “is given explicit information both on-demand and just-in-time, when the [player] needs it or just at the point where the information can best be understood and used in practice” (Gee’s (2003) Explicit Information On-Demand and Just-in-Time Principle as reported in Devlin, 2011, p. 99). As a result, the mathematics may be seen as meaningful within the game world.
6. Regular testing: Players are presented “with frequent tests to see how well they have mastered the latest facts or skills. [...] It’s the *enjoyment* of taking and passing the ‘tests,’ often after several failures, that *motivates* players to learn” (Devlin, 2011, p. 130).

The above ideas have all been carefully implemented in E-Brock Bugs (see Broley et al., in press; Broley, 2013). We hope that the resulting adventure will prompt players to become (better) mathematicians not only in the game world, but in the real world as well.

More About E-Brock Bugs

E-Brock Bugs was created by Laura Broley (2013a), with Chantal Buteau and Eric Muller, as an Honours project in the Mathematics Integrated with Computers and Applications program at Brock University. From the launch of its online version in October 2013 until June 2014, there have been 2475 hits on the game.

From our experience, a Grade 12 student may take anywhere from 90 minutes to 3 hours to save Bug City. A Hall of Fame (n.d.) proudly lists those of Bug City’s heroes who opt to be recognized for their hard work. Be part of it, and encourage your students to be “famous” too! For an unlocked teacher version of the game, visit **www.brocku.ca/mathematics/brock-bugs**.

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The study of mathematics, like the Nile, begins in minuteness but ends in magnificence.
Charles Caleb Colton

The art of doing mathematics consists in finding that special case which contains all the germs of generality.
David Hilbert

▲ A Short Review for a Short Book: *Introductory Calculus for Infants*

Book Review:

(Inouye, O.M. (2011). *Introductory calculus for infants.* Omionline.ca. Available through ThinkGeek and Amazon as listed at **www.omionline.ca/book.html**.)

TIM SIBBALD
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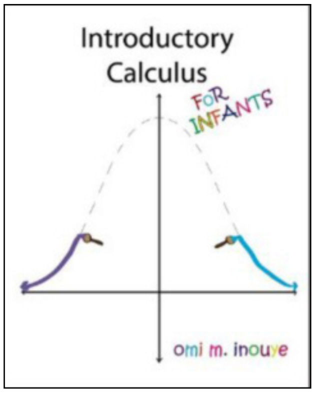

It is rare to come across a book like this where the author notes they are lacking formal qualifications. Ms. Inouye, who is Canadian, notes that she is not a teacher and is an "average math student" (p. 50). What she lacks in formal training she makes up for in creativity. The story begins as an alphabet book but rapidly mentions how "x" does not have any friends and is bullied by the other letters. At this point, "f" comes along and the book moves into being an alphabet book about functions (they are "functional together").

In terms of creativity, this is a colorful book full of doodles that touches on many terminologies that are present in Grades 11 and 12 courses. There are no definitions, but some graphic elements suggest key points. Consider, for example, that there are graphs of the absolute value function, exponential, reciprocal (under "hyperbolic"), step function, sine function, logarithm, and the derivative of a quintic. Domain restrictions also appear, as does a brief mention of the third dimension, convex/concave terminology, and the full mathematical form of the normal distribution.

The place for this inexpensive (<$15) book, in my opinion, is as a light enrichment resource for Grade 11 and 12 students. It is fun and brief, but the few unusual mathematical words may prompt students to investigate further. It is also a reminder that creativity is alive and well, and serves to point to there being room to be inventive within mathematics. That element also makes this book suitable as a gift for math staff members or as an award for deserving students. ▲

▲ Vectors and the Track-and-Field Jumping Events

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Whenever I look at the jumping events in track-and-field, I do not see human bodies in motion; instead (perhaps due to my mathematics background) I see vectors in motion. A vector is an arrow that describes the direction and magnitude of a moving object. Athletes cannot jump higher or further without taking into account Newton's laws of physics. This article will explain how vectors are applied in the various track-and-field jumping events: long jump, triple jump, high jump, and pole vault. More specifically, I will illustrate the impact of horizontal velocity in determining maximum height and distance in the jumps. In all the examples velocity vectors were used; however, they do not consider the application of forces that produce the motion we observe.

One example of vectors can be given with a 100 m sprinter. If an athlete runs toward the finish line with an average velocity of 10 m/s, then he should cross the line in 10 seconds flat. The athlete's velocity vector is represented by Vector A (see Figure 1). If there were a headwind of 2 m/s, as represented by Vector B, then the athlete's motion would be impeded and overall speed would decrease. Vector C represents the resultant vector, whereby the athlete's velocity has decreased to 8 m/s.

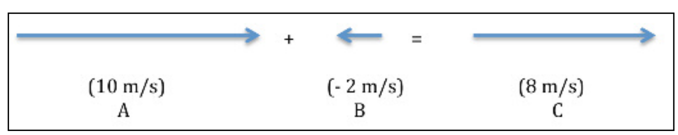

Figure 1: *Athlete's velocity vector with headwind*

Conversely, if there were a tailwind, the athlete would run faster if Vector B were reversed (see Figure 2).

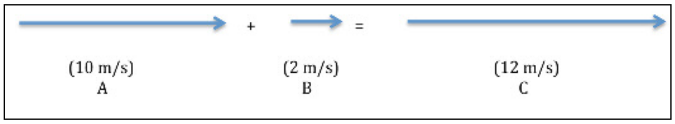

Figure 2: *Athlete's velocity vector with tailwind*

Long Jump

In the long jump, the speed of the athlete before take-off has the greatest impact on absolute distance (Hay, Miller, & Caterna, 1986). Let us consider the following example: a jumper attempts to long jump at a take-off angle of 20 with a take-off velocity of 8 m/s (see Figures 3 and 4). On her second attempt, she has the same take-off angle, but has a take-off velocity of 10 m/s. If all other factors are the same, she will jump further on her second attempt. The horizontal velocity of the jumper combines with the vertical velocity placed at the take-off board to form a resultant vector. In the vertical plane, the force of gravity pulls the jumper toward the ground.

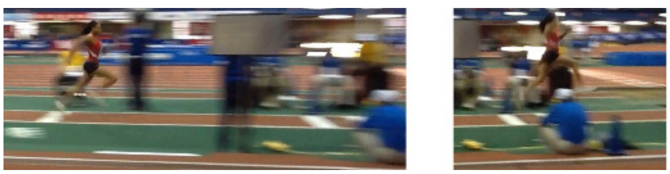

Figure 3: *Athlete approaches the long jump board at 8 m/s, and then takes off*

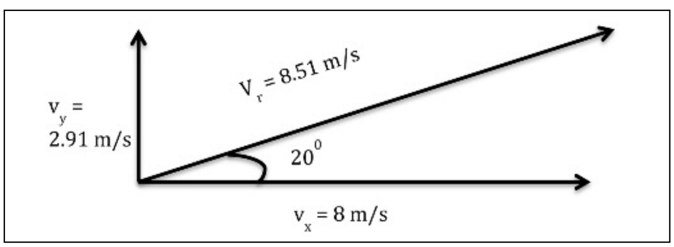

Figure 4. *Vector representation of long-jump take-off.*

Calculations at 8 m/s

Find resultant velocity:

$$\cos\vartheta = \frac{adj}{hyp} \Rightarrow \cos 20 = \frac{8}{r} \Rightarrow r = \frac{8}{\cos 20} \Rightarrow r = 8.51\,\text{m/s}$$

Find velocity of y:

$$\sin\vartheta = \frac{opp}{hyp} \Rightarrow \sin 20 = \frac{y}{8.51} \Rightarrow y = (\sin 20)(8.51) \Rightarrow y = 2.91\,\text{m/s}$$

Find time in the air on the way up: Acceleration due to gravity, g = -9.8 m/s^2

$\Delta v = a \times t \Rightarrow$ $(0 - 2.91) = -9.8t \Rightarrow t = \frac{-2.91}{-9.8} \Rightarrow t = 0.3$

$\therefore$ Total time in the air is 0.6 seconds.

Find distance travelled horizontally:

$d = v + t \Rightarrow d = (8)(0.6) \Rightarrow$ $d = 4.8$ m

Find max height: h = *average velocity* x t

$\Rightarrow$ $h = \left(\frac{2.91+0}{2}\right)$ (0.3) $\Rightarrow$ $h = 0.44$ m

Calculations at 10 m/s

Find resultant velocity:

$$\cos\vartheta = \frac{adj}{hyp} \Rightarrow \cos 20 = \frac{10}{r} \Rightarrow r = \frac{10}{\cos 20} \Rightarrow r = 10.64\,\text{m/s}$$

Velocity of :

$$\sin\vartheta = \frac{opp}{hyp} \Rightarrow \sin 20 = 10.64 \Rightarrow y = (\sin 20)(10.64) \Rightarrow y = 3.64\,\text{m/s}$$

Find time in the air on the way up: Acceleration due to gravity, g = -9.8 m/s^2

$\Delta v = a \times t \Rightarrow$ $(0 - 3.64) = -9.8t \Rightarrow t = \frac{-3.64}{-9.8} \Rightarrow t = 0.37$

$\therefore$ Total time in the air is 0.74 seconds.

Find distance travelled horizontally:

$d = v + t \Rightarrow d = (10)(0.74) \Rightarrow$ $d = 7.4$ m

Find max height: h = *average velocity* x t

$\Rightarrow$ $h = \left(\frac{3.64+0}{2}\right)(0.37) \Rightarrow$ $h = 0.67$ m

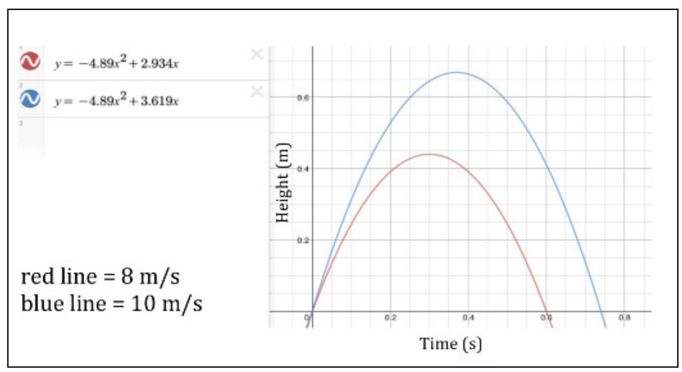

Figure 5: *Graph of parabolas' height versus time*

Based on this example, it is evident that athletes who wish to jump further in the long jump should work on increasing the amount of speed they generate before take-off. This in turn will produce longer time in the air, greater height off the board, and further jump distance (see Figure 5).

Triple Jump

Elite triple jumpers achieve an optimal speed that is slightly less than their maximal speed because it is very difficult to hop and bound at full speed. There are three phases in the triple jump (the hop, step, and jump), and during each phase, the athlete is experiencing some deceleration (Hay & Miller, 1985). The athlete is attempting to travel the greatest distance possible during each phase by applying maximum forces to the ground (see Figure 6). One common error by triple jumpers is to come in with low runway speed and attempt to maximize the forces of each of the three ground contacts in the jump. This is a losing battle because, as seen in the long-jump calculations, greater horizontal velocity produces greater horizontal displacement.

Figure 6: *The hop, step, and jump phases, respectively, of the triple jump event*

Some athletes may choose to be hop-phase dominant, while others may be jump-phase dominant. Consider the following example that compares the two styles of triple jump. Hop-phase dominant (Figure 7) shows a large hop followed by a shortened step and almost equivalent jump phase. There is more deceleration between phases, since flight is greater on the hop phase. The overall distance of the jump with the three combined phases is 12.50 m.

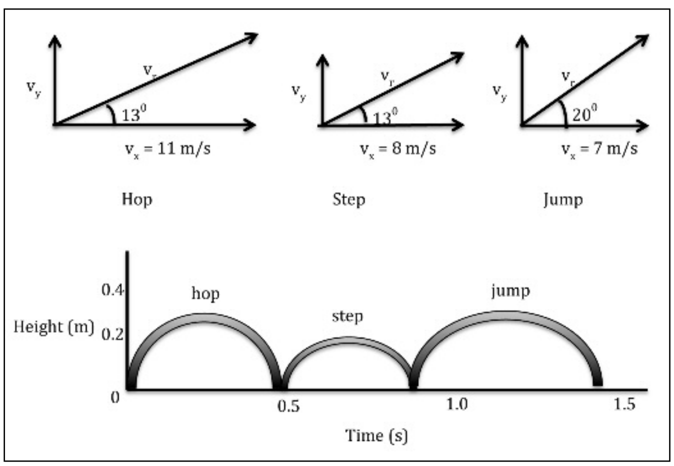

Figure 7: *Vector and parabolic representations of the hop-phase dominant form of triple jump*

Jump-phase dominant (Figure 8) shows a shortened hop followed by a shortened step and large jump phase. There is less deceleration between phases, since the take-off angle of the hop phase is minimized. The overall distance of the jump is also 12.50 m.

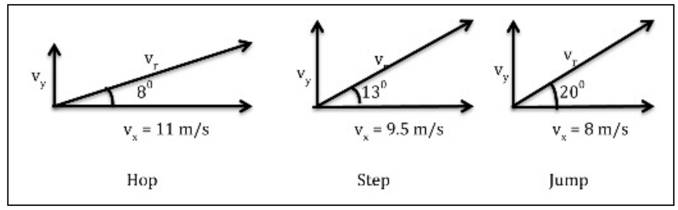

Figure 8: *Vector representation of jump-phase dominant form of triple jump*

High Jump

Generally high jumpers are classified as either "power jumpers" or "speed jumpers" based on runway speed. There are many factors (which are beyond the scope of this paper) that need to be considered to fully understand the mechanics of the high jump, including centripetal forces, mass, height, and centre of gravity over the bar. Consider the following example that uses vectors (see Figure 9): an athlete attempts to clear a height with an incoming horizontal speed of 2 m/s with a take-off angle of 70. The same athlete attempts to clear the same height with an incoming speed of 2.2 m/s at the same take-off angle. According to the calculations for both jumps at 2 m/s, her maximum clearance is 1.56 m, and at 2.2 m/s, her maximum clearance is 1.87 m.

Figure 9: *Visual and vector representations of high jump athlete at take-off*

This example shows that subtle differences in horizontal velocity can impact the height of a projectile when the angle of take-off approaches the normal line that is perpendicular to the jumping surface. The athlete will need to take off further away from the bar to safely clear the bar at the apex of the jump.

Pole Vault

The pole vault requires a transfer of energy into a fibreglass or carbon-fibre pole that bends while propelling the athlete upwards over a bar. Pole vault is the only event in track and field that is limited by the equipment the athlete possesses. In essence, the bigger the pole the athlete can bend, the higher he or she can jump. I consider the pole-vault the most technical of the jumping events. For the purpose of this paper, the pole vault event will only be discussed as far as loading the pole with elastic energy. Pole vault is very similar to long jump in the sense that the athlete is attempting to jump up to the bar with as much horizontal speed as possible (see Figure 10).

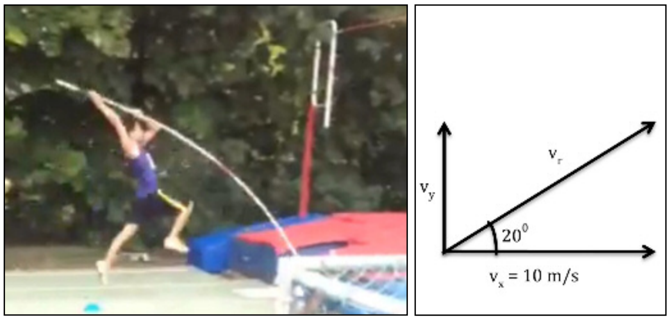

Figure 10: *Visual and vector representations of the pole vault athlete at point of take-off*

The greater the speed, the more kinetic energy available to be stored as elastic energy when the pole bends (see Figure 11).

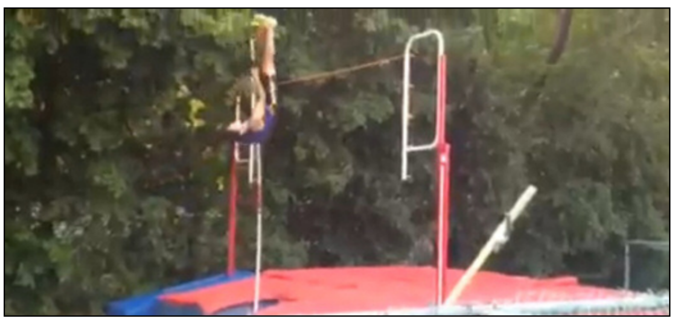

Figure 11: *Recoil of the pole launching the pole vault athlete upwards*

Great vaulters have the right combination of athleticism, gymnastic ability, and fearlessness. Technique plays an important role in jumping high, but at an international level, the athletes all possess tremendous runway speed. Olympic-level athletes are able to transfer their speed effectively to the pole for highest vertical propulsion. Subtle technical errors often distinguish the medalists from non-medalists.

Conclusion

Throughout this article, there has been one constant theme for all the jumping events: speed and the transfer of speed are the most important elements of success. I used vectors to represent bodies in motion on the runway, as well as vertical forces on take-off. The resultant vectors create parabolic pathways for the human body in each of the jumping events. Teachers can use some of these examples to create real-life scenarios for vector problems. Calculations of horizontal distance and maximum height are given. Further extension questions may include calculations of parabolic equations for distance and height, investigations with changes in speed and angles for high jump and long jump, and analyzing the effectiveness of hop-phase dominant and jump-phase dominant triple-jump technique. Common technical errors were also discussed for the various jumping events. I am aware that this paper is only a simplified version of all the jumping events, and that many other technical factors such as running technique, braking forces, centre of mass, cross winds, core instability, and poor reactive strength were not examined.

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▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Lynda Colgan, Queen's University

Projects: Of interest to the OAME community (and in particular, our Elementary members) may be the fact that Lynda has been contracted by CODE (The Council of Ontario Directors of Education) and the Ontario Ministry of Education to develop a parent toolkit for mathematics education. This toolkit, due to be launched in April 2015 at the annual provincial parent involvement conference at the Ministry of Education, will contain the following elements: (i) a print resource (separate chapters for Kindergarten, and Primary, Junior, and Intermediate Divisions, and General Parent Involvement); (ii) videos for the five modules described above; and (iii) an Implementation Guide with workshop plans, blackline masters, and fact sheets for all five modules. This toolkit has been informed by the work that she has been doing with participants at the provincial EQAO Parent Forums over the past two years, and her work with Parent Involvement Committees (PICs) across Ontario. ▲

▲ Recasting Mad Minutes: Going Back to the Basics?

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Marc Husband is seconded faculty at York University from the Toronto District School Board. Tina Rapke is a faculty member at York University, who is jointly appointed to the Faculties of Education and Science. Marc Husband and Tina Rapke can often be seen co-planning and co-teaching with each other and school teachers.

Introduction

We set up a research study that examined a recasting (implementation) of mad minutes to fit within the current curriculum because of the attention mathematics education has been receiving in the media. In particular, we wanted to speak to parents and other groups' petitions for curriculum changes and pushes for "back to the basics" (Marrow, 2014). Our inspiration came from Carson's (2014) article about the "math wars," in which she pointed to the disappearance of the mad minutes parents experienced in school. As mathematics educators, this attracted our attention. We want parents and students to be able to communicate and understand each other through common experiences. We feel that the task plays an important role, but it is secondary to how a task is implemented and unfolds. To address these concerns, we developed a research project that incorporated the Ontario Mathematics curriculum's emphasis on student-generated strategies (see, for example, pages 44, 56, 67, 79, 122) into a lesson involving mad minutes.

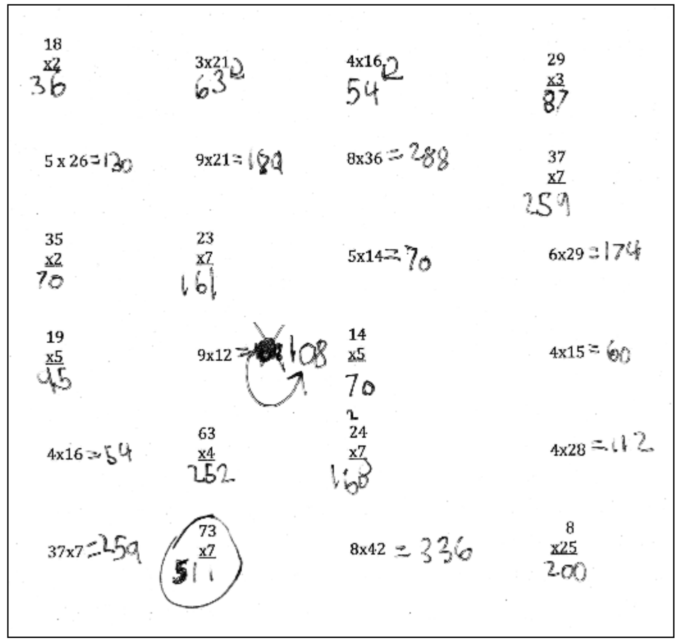

Figure 1: *Mad-minute worksheet*

Study design

We chose to have a small sample size so that we could qualitatively analyze and report on students' understandings through a re-casting of mad-minutes. The classes in which the recasting of mad minutes took place were co-planned and co-taught by the researchers and the teacher. Although we acknowledged the limitations of a statistical study based on a small sample size, we felt it was appropriate to collect quantitative data because the "math wars" discussions have had a strong grounding in test scores. We attempted to address some limitations, in particular, demographic issues, by conducting the study in one school.

Participants

The study took place within an English public school located in Toronto. Over two-thirds of the school's student population came from English-speaking backgrounds. Our research involved two Grade 4 classes. There was a treatment group who experienced the recasting of mad minutes, and a control group who were previously taught the standard algorithm. There were two IEP volunteer research participants in the control group class. While the treatment group volunteer participants included five Individual Education Plan (IEP) students and one English Language Learner (ELL) student. The teachers involved in the study seemed to have different focuses. The control group teacher explained that she had previously taught and introduced one-digit by two-digit multiplication in a rote fashion by placing emphasis on the steps involved in the traditional multiplication algorithm. The treatment group teacher told us that she valued number sense and that this value came from her literacy training. The treatment group teacher also explained that she and her students had not previously discussed one-digit by two-digit multiplication.

Structure of the days when the study took place

Both groups were initially given a mad-minute worksheet that consisted of 20 one-digit by two-digit multiplication questions, and students were asked to complete as many questions as they could in 20 minutes. Many of us have had first-hand experience with this type of math lesson. You may imagine dreary math experiences with individual worksheets checked for correctness. After students completed their individually timed mad minutes, the treatment group experienced a different type of mathematics lesson. (See Table 1.)

The treatment group students experienced a recasting of mad minutes in two ways. First, instead of a focus on right or wrong solutions, students were invited to identify “easy” and “hard” questions and to explain their thinking. Second, students created mad-minute questions for their classmates to complete based on their generated multiplication strategies. The research project lasted seven school days.

Students generating strategies and describing their thinking

Within a few minutes of being asked to share their thinking about an easy question, a strategy that involved the distributive property, a(b+c)=ab+ac, emerged within two different multiplication questions: 5x14 and 4x28. Students told us that they converted these questions to addition questions. They said that 5x14 is 5 times 10 plus 5 times 4, which is 50 plus 20 and 4x28 is 80+32. As the discussion progressed, it became evident that the students were familiar with some facts and referred to these facts as “known facts.” Known facts penetrated other strategies as students wove what they already knew before whole-group discussion and newly constructed strategies being elicited by the discussion. For example, a student said that he used doubles to solve 8 x 25 because for him, 4 x 25 is a “known fact,” and another student said he knew 4 x 25 = 100 because of money.

Eventually, the class came to an agreement that 18 x 2 was one of the “easier” questions to solve. In the beginning, one student reasoned that it was easy because the numbers were small, but once the teacher prompted further, more strategies emerged. Skip counting by 2’s 18 times, and skip counting from 20 eight times, were both shared and recorded. When the teacher asked the group if there were any other strategies or ways we could think about 18 x 2, a student raised her hand. She explained to the class that she took 18 and split it into two to get 9, and then multiplied the 2 by 2 to get 4. She declared that she had changed 18 x 2 to 9 x 4. Another student shared a doubling strategy with the class on a “hard” one. She explained to the class that she used a doubling strategy for 37 x 7. She doubled 37 to get 74, doubled again to get 148, and declared that this was 4 times 37. The student continued to double 148 to get 296, and because she knew that she’d doubled and had actually computed 37 x 8, she subtracted 37 to get 259. See Figure 2 for a representation and summary of students’ thinking and generated strategies.

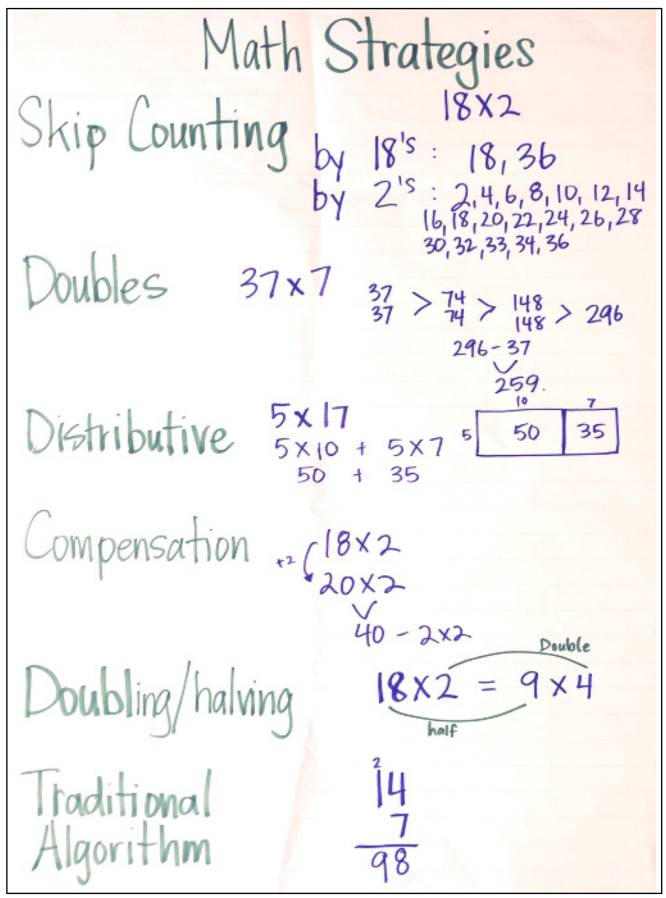

Figure 2: *Generated strategies*

Statistics and student samples

See Table 2 to view the statistics about the mad-minute results. It is interesting to note that the treatment group's average went down after the recasting of mad minutes. Initially, we thought that the post-test may have been "harder"; however, a comparison with the treatment group's average indicated otherwise. We predicted the decrease in average was because it takes time to think about the most appropriate strategy for each individual mad-minute question. It was felt that students needed more encounters and practice with one-digit by two-digit multiplication questions and their generated strategies. Our prediction was confirmed by having the treatment group student write two additional mad-minute exercises and by observing a significant increase in the class average.

	Pre-test	Post-test 1	Post-test 2	Post-test 3
Control group averages	83%	85%	n/a	n/a
Treatment group averages	66%	53%	72%	78%

Table 2: *Control group and treatment group averages of mad-minute worksheets*

We also noticed a difference between the treatment and control groups' explanation of their thinking. The students in the treatment group provided explanations of their thinking that provided evidence of them making sense of mathematics, whereas the control group, for the most part, presented us with computations using the standard algorithm. See Figure 3 and Figure 4.

Conclusion/Discussion

We hope that reading about how we implemented mad minutes to fit within the current curriculum exposes the difference between the ways in which parents may have learned in school and how students, whose experiences are grounded in current curriculum, are learning mathematics. We would also like to note that the recasting fostered challenge. It was fascinating to witness that, within the recasting of mad minutes, students seemed to crave challenge. For example, the control group teacher had to prompt several times for students to share the "easier" questions, as students seemed to be excited to talk about the "hard" ones. Also, when students were asked to include a "tricky" question in the mad-minutes exercises they created, most of the groups posed the "tricky" multiplication question first, seeming to be eagerly wanting to challenge their peers.

	DAY 1	DAY 2	DAY 3	DAY 4	DAYS 5, 6, & 7
Treatment group	• Mad-minute pre-test. • Students generated multiplication strategies.	• Students made their own "mad-minute" questions.	• Students solved each other's "mad-minute" questions.	• Students received their peers' completed "mad minutes" and discussed the solutions. • Some students decided to change their initial strategy because they perceived their peers' strategy to be quicker and/or more efficient.	• Students solved three mad-minute post-tests.
Control group	• Mad-minute pre-test.		• Students solved treatment groups' generated "mad-minute" questions. • They were asked to explain their thinking.		• Students solved one mad-minute post-test.

Table 1: *Activities for each day of the study*

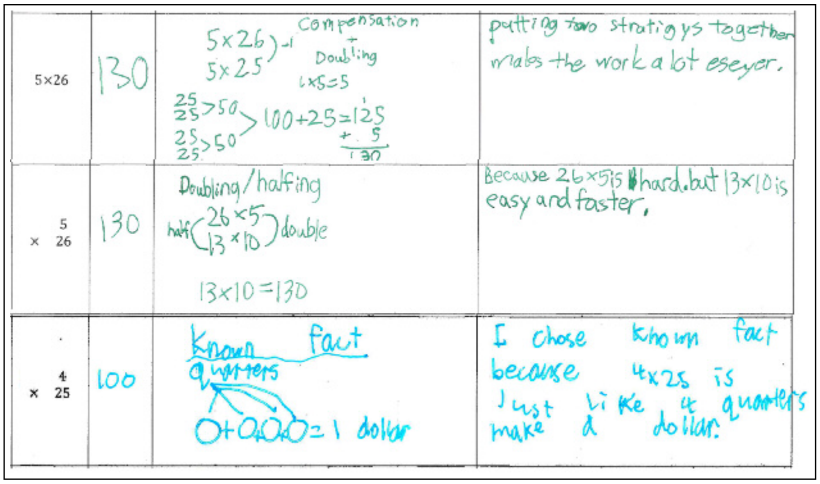

Figure 3: *Samples of three students' work from the treatment group*

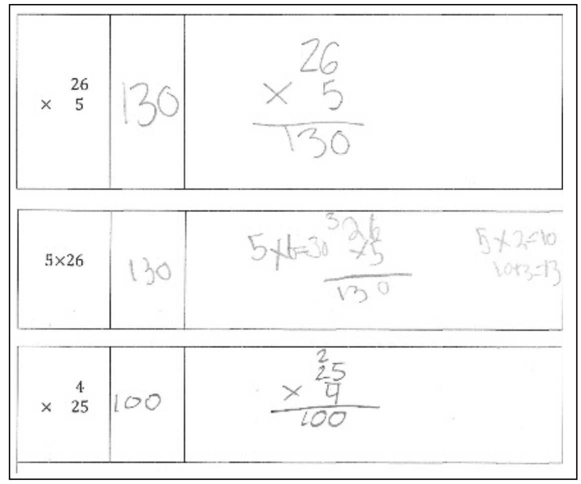

Figure 4: *Samples of three students' work from the control group*

We must also mention that some of the students from the treatment group declared that they could use the distributive property to solve any of the questions, i.e., 21 x 5 = 20 x 5 + 1 x 5 = 100 + 5. A comparison of this thinking and the computation of 21 x 5, using the standard algorithm,

$$\begin{array}{r} 21 \\ \underline{x5} \\ 5 \\ \underline{+100} \\ 105 \end{array}$$

reveals that the distributive property is at the heart of the standard multiplication algorithm. We feel that the students in the treatment group, through participating in recasting mad minutes, were primed to make sense of why the standard multiplication algorithm works.

Furthermore, students in the treatment group were learning mathematics with understanding. Hiebert et al. (1997) say that "we understand something if we see how it is related or connected to other things we know" (p. 4). Students related and connected mad-minute questions and their generated strategies to their prior knowledge by converting multiplication questions to addition questions and solving and generating strategies that used their known facts. In the end, the recasting of mad minutes drew out and built on students' prior knowledge and brilliant thinking. Students were making sense of mathematics.

Although we are aware of the limitations of conducting statistical analysis with small sample sizes, we feel that our study design choice of collecting quantitative data, in addition to the qualitative data, proved to be fruitful. The test averages informed our qualitative predictions, and on a whole, allowed us to provide a richer narrative that speaks to procedural fluency and provokes further departures that could lead to additional productive research and discussions about the "math wars." We feel that the real question that recasting mad minutes has left us to ponder as teachers, mathematics educators, and parents is: Do we want students to build on prior knowledge so that they can learn mathematics with understanding, or do we simply want them to memorize standard algorithms like most of us did?

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It is the supreme art of the teacher to awaken joy in creative expression and knowledge.
A. Einstein

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