MHF 4U Unit 2 –Rational Functions– Outline

Day	Lesson Title	Specific Expectations
1 (Lesson Included)	Rational Functions and Their Essential Characteristics	C 2.1,2.2, 2.3
2 (Lesson Included)	Rational Functions and Their Essential Characteristics	C 2.1,2.2, 2.3
3 (Lesson Included)	Rational Functions and Their Essential Characteristics	C 2.1,2.2, 2.3
4	Rationale Behind Rational Functions	C3.5, 3.6, 3.7
5 (Lesson Included)	Time for Rational Change	D1.1- 1.9
6-7	JAZZ DAY	
8	SUMMATIVE ASSESSMENT	
TOTAL DAYS:		8

Unit 2: Day 1: Rational Functions and Their Essential Characteristics		MHF 4U1
Minds On: 5 Action: 55 Consolidate:10 Total=75 min	**Learning Goal**: Students will • Investigate and summarize the characteristics (e.g. zeroes, end behaviour, horizontal and vertical asymptotes, domain and range, increasing/decreasing behaviour) of rational functions through numeric, graphical and algebraic representations.	**Materials** • BLM 2.1.1, BLM 2.1.2, BLM 2.1.3 • Graphing calculators

		Assessment Opportunities
Minds On…	**Whole Class → Discussion** Engage students in a discussion by asking them to respond to the following prompts: • What is a rational function? • Compare and contrast rational and polynomial functions • What might be the restrictions on rational functions?	
Action!	**Pairs → Investigation** Students will have further opportunity to reflect on these questions as they complete BLM 2.1.1. **Whole Class → Discussion** Have students summarize their results from BLM 2.1.1. Referring to BLM 2.1.2, ensure that the key points of rational functions are highlighted. Using the questions provided, engage the students in an exploration of the key characteristics of rational functions. **Curriculum Expectations/Observations/Mental Note** Identify students' prior knowledge throughout the investigation and discussion. **Mathematical Process: Reasoning** Students will reason as they explore key characteristics of rational functions.	
Consolidate Debrief	**Whole Class → Note** Have the students refine their summary to complete their note.	
Exploration *Application*	**Home Activity or Further Classroom Consolidation** Complete BLM 2.1.3	

BLM 2.1.1: Scuba Diving

Scuba divers must not hold their breath as they rise through water as their lungs may burst. This is because the air, which they have breathed to fill their lungs underwater, will expand as the scuba diver rises and the pressure on the body reduces. At every depth, the diver wants 6 litres of air in her lungs for breathing.

If a diver holds her breath, the volume of the air in her lungs varies with the pressure in the following manner:

Volume (at new pressure) = $\frac{originalvolumeXoriginalpresure}{newpressure}$

The pressure is 1 atmosphere at the surface and increases by 1 atmosphere for every 10 metres below the surface.

1. A diver takes a 6-litre breath of air at the surface and descents without breathing. Using the formula above, complete the following table.

Depth (D) in metres	0	10	20	30	40	50	60
Pressure (P) in atmospheres	1	2	3				
Volume (V) of air in lungs in litres	6	2					

2. (a) A diver takes a 6-litre breath of air from her tank at 60 metres. Imagine that she can ascent without breathing. Complete the following table.

Depth (D) in metres	0	10	20	30	40	50	60
Pressure (P) in atmospheres	1	2					
Volume (V) of air in lungs in litres							6

(b) Find the rule connecting P and D
Check the rule for D = ,0 and 40 and 50

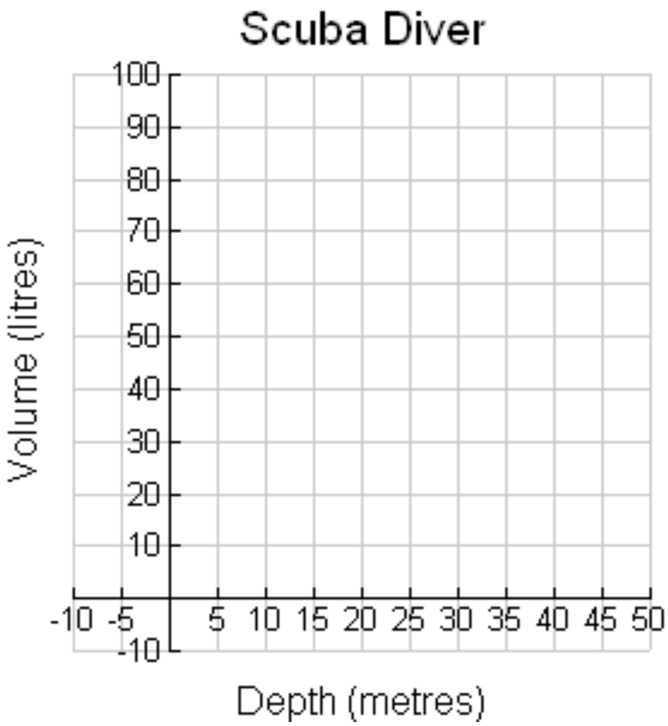
(c) Find the rule connecting V and P
Check this rule for P = 3, 4 and 5.

(d) Use algebra to show the V = $\frac{280}{D+10}$

(e) Use your graphing calculator to graph the volume of air against depth of the diver in metres.

BLM 2.1.1: Scuba Diving **(continued)**

(f) Sketch your graph on the grid provided.



(g) What happens to the graph as the depth increases?

(h) What happens to the graph as the depth moves into the negative values?

BLM 2.1.2: Essential Characteristics of Rational Functions

A **rational function** has the form h(x) = $\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomials

The domain of a rational function consists of all real number except the zeroes of the polynomial in the denominator. g(x) 0

The zeroes of h(x) are the zeroes of f(x) if h(x) is in simplified form.

1. What is the domain of each rational function?
   Determine the x- and y-intercepts. Then graph y = f(x) with graphing technology and estimate the range.

   (a) $f(x) = \frac{5}{x-2}$ (b) f(x) = $\frac{x}{x^2-2x-3}$ (c) f(x) = $\frac{2x^2+5x-3}{x^2-4}$

2. Find the vertical and horizontal asymptotes of g(x) = $\frac{x+5}{x^2-4}$.

   Vertical: Find where the function is undefined.

   Horizontal: Set up a chart for large negative and positive values of x.

   E.g.

x	g(x)
100	
500	
1000	
2000	
5000	
10000	
100000	
700000	

3. Let f(x) = $\frac{x}{x^3-2x^2-5x+6}$. Find the domain, intercepts, and vertical and horizontal asymptotes. Then use this information to sketch an approximate graph.

BLM 2.1.3: Rational Functions and Their Essential Characteristics

For question 1 – 6, refer to the following functions.

(a) $f(x) = \dfrac{x+3}{x-5}$ (b) $g(x) = \dfrac{x-2}{x^2+7x+10}$

(c) $h(x) = \dfrac{x^2-3x+2}{x+2}$ (d) $k(x) = \dfrac{x^2-4}{x^3+-2x^2-5x+6}$

1. Find the x- and y-intercepts of each function.
2. Write the domain for each function.
3. Find the vertical asymptote(s)
4. Find the horizontal asymptote(s)
5. Use the information from questions 1 to 5 to graph each function.
6. Check by using graphing technology.
7. Functions R(x) = $-2x^2 + 8x$ and C(x) = 2x+1 are the estimated revenue and cost functions for the manufacture of a new product. Determine the average profit function AP(x) = $\dfrac{P(x)}{x}$. What is the domain of AP(x)? When is the Profit equal to zero?
8. Repeat question 7 for R(x) = $-x^2 + 20x$ and C(x) = 7x+30.
9. The model for the concentration y of caffeine the bloodstream, h hours after it is taken orally, is $y = \dfrac{6h}{h^2+1}$. What is the domain of y in this context? Graph the function. What is the concentration of caffeine after 12 hours?
10. A rectangular garden, 40 m^2 in area, will be fenced on three sides only. Find the dimensions of the garden to minimize the amount of fencing.
11. What is a rational function? How is the graph of a rational function different from the graph of a polynomial function?
12. For each case, create a function that has a graph with the given features.
    (a) a vertical asymptote x = 2 and a horizontal asymptote y = 0
    (b) two vertical asymptotes x = -2 and x = 1, horizontal asymptote y = -1, and x-intercepts – 1 and 3.

BLM 2.1.3: Rational Functions and Their Essential Characteristics (continued)

Answers:

1. (a) x-int (-3,0), y-int (0,-3/5) (b) x-int (2,0), y-int: (0,-1/5)
   (c) x-int: (2,0), (1,0), y-int: (0,1) (d) x-int(2,0),(-2,0), y-int: (0,-2/3)

2. (a) x□5 (b) x□-5,-2
   (c) x□-2 (d) x□1,-2,3

3. (a) x =57 (b) x = -5, x = -2
   (c) x = -2 (d) x = 1,-2,3

4. (a) y = 1 (b) y=0 (c) none (d) y = 0

5. Graphs

6. Graphing Calculator

7. AP(x) = $\dfrac{-2x^2+7x-6}{x}$ or $-2x+6-\dfrac{6}{x}$, D: x>0. Break even: x = ½,2 (zeroes)

8. AP(x) = $\dfrac{-(x-3)(x-10)}{x} or -x+13-\dfrac{30}{x}$. D: x>0, Break even: x= 3 or 10 (Zeroes)

9. D: h∃0. Functions increases to a maximum of (1,3) After 0.497 Model is reasonable.

10. 10m X 20 m

12. (a) $y=\dfrac{1}{x-2}$ (b) $y=\dfrac{-x^2+2x+3}{x^2+x-2}$

Unit 2: Day 2: Rational Functions and Their Essential Characteristics		MHF 4U1
Minds On: 15 Action: 55 Consolidate:10 Total=75 min	**Learning Goal**: Students will • Demonstrate an understanding of the relationship between the degrees of the numerator and the denominator and horizontal asymptotes • Sketch the graph of rational functions expressed in factored form, using the characteristics of polynomial functions.	**Materials** • BLM 2.2.1 • BLM 2.2.2 • BLM 2.2.3
		Assessment Opportunities
Minds On…	**Pairs → Activity** Display BLM 2.2.1 for the students. Have them work in pairs to determine the asymptotes. **Whole Class → Discussion** Have students share their understanding of asymptotes. **Curriculum Expectations/Observation/Mental Note** Assess students' ability to determine horizontal and vertical asymptotes.	
Action!	**Whole Class → Discussion** Engage the students in a discussion about asymptotes by referring to BLM 2.2.2. **Mathematical Process: Connecting** Students will connect their prior knowledge of algebraic manipulation to the determination of asymptotes.	
Consolidate Debrief	**Whole class → Discussion** Have students summarize their understanding of oblique asymptotes in a note by responding to the following prompts: • Discuss how to identify if a rational function will have an oblique asymptote • Describe the type of oblique asymptote	
Practice Application	**Home Activity or Further Classroom Consolidation** Complete BLM 2.2.3.	

BLM 2.2.1: Name That Asymptote

For the following functions name the vertical and horizontal asymptotes

1. $f(x) = \dfrac{x}{3x-6}$

2. $g(x) = \dfrac{x-4}{x^2-5x+6}$

3. $h(x) = \dfrac{3x^2-6x}{x^2-1}$

BLM 2.2.2: All About Asymptotes

1. Discuss what happens to the value of the function $f(x) = \frac{1}{x}$ as x→+4 and x→-4

2. A photocopying store charges a flat rate of $1 plus $0.05/copy.
   (a) Write a function f(x) to represent the average cost per copy.
   (b) Determine what happens to the function as x becomes very large.

3. Find the horizontal asymptote for $g(x) = \frac{2x^2 - 3x + 1}{1x^2 - 4x + 5}$

4. Oblique Asymptotes
   For rational function, linear oblique asymptotes occur when the degree of the numerator is exactly one more than the degree of the denominator. The equation of the linear oblique asymptote can be found by dividing the numerator by the denominator.

   Determine the oblique asymptote for y = $\frac{3x^3 - 2x^2 + 5}{x^2}$

   Use the oblique asymptote and the vertical asymptote to sketch the graph.

BLM 2.2.3: Asymptotes of Rational Functions

1. Find the equation of the horizontal asymptote of each curve.

   (a) $f(x) = \dfrac{2x+3}{x+5}$ (b) $g(x) = \dfrac{x}{x^2-9}$

   (c) $h(x) = \dfrac{x^2-1}{x^2+1}$ (d) $k(x) = 1 - \dfrac{x}{x^2-4}$

2. Find an equation of the oblique asymptote of each curve.

   (a) $y = \dfrac{2x^2-4x+6}{x}$ (b) $h(x) = \dfrac{x^3-8}{x^2}$

   (c) $y = \dfrac{9x^2}{3x-1}$ (d) $f(x) = \dfrac{x^3+6x^2+9x+10}{x^2+4}$

3. Find the linear oblique asymptote of each curve and use it to help you sketch the graph. Use a graphing calculator to check your result.

   (a) $y = 2x + 1 + \dfrac{3}{x+2}$ (b) $y = \dfrac{x^2-9}{x-5}$

4. A robotic welder at General Motors depreciates in value, D, in dollars, over time, t, in months. The value is given by $D(t) = 10000 - \dfrac{4000t}{(t+2)}$.

   (a) Find the value of the machinery after:
   (i) 1 month (ii) 6 months (iii) 1 year (iv) 10 years

   (b) Would you have a local maximum or local minimum in the interval [0,4]? Explain.

   (c) Find the D(t) is t becomes extremely large.

   (d) Will the machinery ever have a value of $0?

   (e) In light of your result in part (d), does V(t) model the value of the machinery for all time?

5. For the function $f(x) = \dfrac{x^2+5x+6}{x-2}$, use the domain, intercepts and vertical, horizontal and oblique asymptotes to sketch the graph.

6. Empire Flooring installs hardwood flooring and charges \$500 for any area less than or equal to 30 m^2 and an additional \$25/$m^2$for any are over 30 m^2.
   (a) Find a piecewise function y = c(x) to represent the average cost, per square metre, to install s square metre of carpet.
   (b) Find the value of c(x) as x becomes extremely large
   (c) Graph y = c(x) for x>0
   (d) Would it be economical to have this company install hardwood for an area of 5 m^2 ? Explain.

7. (a) Under what conditions does a rational function have a linear oblique asymptote?
   (b) Explain how to find the linear oblique asymptote of a rational function

8. Bell Canada's sales for the last 10 years can be modelled by the function $S(n)=\dfrac{1n^2+2n+4}{15n+4}$, where S(n) represents annual sales, in millions of dollars, and n represents the number of year since the company's founding.
   Find S(n) as n becomes extremely large. Interpret this result

Answers:

1.	(a) y = 2		(b)	y = 0		(c)	y = 1	(d)	y = 1
2.	(a) y = 2x-4		(b)	y=x		(c)	y = 3x+1	(d)y = x + 6	
3.	(a) y = 2x + 1			(b)	y = x + 5				
4.	(a)	(i) \$8667.67	(ii) 7000		(iii) \$6571.43		(iv) 6065.57		
	(b) max at t = 0								
	(c) 6000		(d)	no		(e)	no		
5.	Graph								
6.	(a) $C(x)=\dfrac{500}{x} x\le 30$, $C(x)=\dfrac{500+25x}{x}, x>30$								
	(b) 525								
	(a) No								
8.	(a) S(n) = 4								

Unit 2: Day 3: Rational Functions and Their Essential Characteristics		MHF 4U1
Minds On: 5 Action: 55 Consolidate:15 Total=75 min	**Learning Goal**: Students will • Investigate and summarize the characteristics (e.g. zeroes, end behaviour, horizontal and vertical asymptotes, domain and range, increasing/decreasing behaviour) of rational functions through numeric, graphical and algebraic representations. • Solve rational inequalities, where the enumerator and denominator are factorable • Approximate the graphs of rational functions and use this information to solve inequalities	**Materials** • BLM 2.3.1 • BLM 2.3.2
		Assessment Opportunities
Minds On…	**Whole Class → Discussion** Have students consider the following questions: • What is the difference between an equation and an inequality? • How would we solve a rational inequality? e.g. $\frac{x}{x+7} < \frac{-x}{x-2}$ Have students generate a solution to the rational inequality. **Mathematical Process: Problem Solving, Reasoning** Students will connect their prior knowledge of rational numbers and rational functions to solving a rational inequality.	
Action!	**Whole Class → Lesson** Engage the students in a discussion about asymptotes by referring to BLM 2.3.2. **Curriculum Expectations/Observation/Mental Note** Assess students' ability to graph rational functions. .	
Consolidate Debrief	**Whole Class → Discussion** Students will create a class note which requires them to • Summarize the characteristics of a rational functions • Discuss how to find zeroes, end behaviour, asymptotes etc of a rational function • Discuss how to solve a rational inequality	
Exploration *Application*	**Home Activity or Further Classroom Consolidation** . Complete BLM 2.3.2	

BLM 2.3.1: Solving Inequalities

Definition of a Rational Function

A **rational function** has the form h(x) = $\dfrac{f(x)}{g(x)}$, where f(x) and g(x) are polynomials

The domain of a rational function consists of all real number except the zeroes of the polynomial in the denominator. g(x) $\neq$ 0
The zeroes of h(x) are the zeroes of f(x) if h(x) is in simplified form.

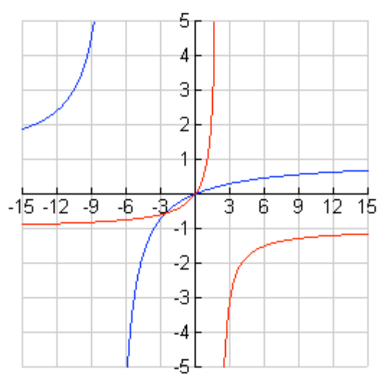
1. Let f(x) = $\dfrac{(x+3)(x-2)^2}{x^3(x+1)}$. Find the domain, intercepts, and vertical and horizontal asymptotes. Then use this information to sketch an approximate graph.

2. Solve $\dfrac{x}{x+7} < \dfrac{-x}{x-2}$ by graphing the functions $f(x) = \dfrac{x}{x+7}$ and $g(x) = \dfrac{-x}{x-2}$

   (a) Find the intersection points of the two graphs. Determine when f(x) is less than g(x) from the graph.

   (b) Find the intersection point(s) algebraically.

f g



BLM 2.3.2 Graphing Rational Functions and Solving Inequalities

Answer questions 1 to 5 without graphing technology.
For question 1 – 6, refer to the following functions.

(a) $f(x) = \dfrac{x}{2x-10}$ (b) $g(x) = \dfrac{3x^2-5x+2}{x}$

1. What are the x- and y-intercepts of each function?
2. Write the domain for each function.
3. What are the vertical asymptote(s)?
4. What are the horizontal asymptote(s)?
5. Use the information from questions 1 to 5 to graph each function.
6. Check your answer using a graphing calculator.
7. Use the domain and intercepts, and vertical, horizontal and oblique asymptotes to graph each function.

   (a) $f(x) = \dfrac{2x+1}{x^2-2x-3}$ (b) $g(x) = \dfrac{3x^2-7x}{x^2-1}$ (c) $h(x) = \dfrac{x^2-1}{x^3-2x^2-x+2}$

8. Solve the following rational inequality. Use a graphing calculator to find the intersection

$$\frac{x}{2x-8} \geq \frac{x^2+x-6}{x+2}$$

9. What is a rational inequality? How do you solve a rational inequality?
10. For each case, create a function that has a graph with the given features.
    (b) a vertical asymptote x = 3, a horizontal asymptote y =0, no x-intercepts, and y-intercept = -1
    (b) a vertical asymptote the y-axis, an oblique asymptote y = 2x+1 and no x- and y-intercepts.

Answers:

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1. (a) x-int=0,y-int=0 (b) x-int=2/3,1, y-int=none
2. (a) $x \neq 5$ (b) $x \neq 0$
3. (a) x = 5 (b) x = 0
4. (a) y = 0 (b) no
8. x#-2.95, -2<x#1.71, 4<x#4.75
10. (a) $y = \dfrac{3}{x-3}$ (b) $y = \dfrac{2x^2+x+1}{x}$

Unit 2: Day 5: Time for Rational Change		MHF4U
Minds On: 5 Action: 50 Consolidate:20 Total=75 min	**Learning Goal**: Solve problems involving average and instantaneous rates of change at a point using numerical and graphical methods Investigate average rates of change near horizontal and vertical asymptotes	**Materials** BLM 2.5.1
		Assessment Opportunities
Minds On…	**Small Groups → Investigations** Work in groups to solve one problem listed on BLM 2.5.1 Discuss major points of their investigation	
Action!	**Small Groups → Presentation** Present solutions to their problem to the class	
Consolidate Debrief	**Whole Class → Discussion** Discuss major points of investigations and any conclusions that can be drawn. Ask questions for clarification	
	Home Activity or Further Classroom Consolidation .Complete remaining problems on BLM 2.5.1	

2.5.1 Rate of Change Problems

1. The electrical current in a circuit varies with time according to $c = \dfrac{3s^3 - s^2 + 5s}{s^3 + 10}$, where the current, ***c***, is in amperes, and time ***s*** is in seconds. Find the average rate of change from 0.75 seconds to 1.5 seconds, and find the instantaneous rate of change at 1.5 seconds. Identify any vertical asymptotes.

2. As you get farther from Earth's surface, gravity has less effect on you. For this reason, you actually weigh less at higher altitudes. A person who weighs 55kg can use the function's $W(h) = \dfrac{6400(55)}{h + 64000}$ to find their weight, ***W*** in kgs, at a specific height, ***h*** in feet above sea level, above the Earth's surface. Find the average rate of change from heights of 750 ft to 1200 ft above sea level, and find the instantaneous rate of change at 1200 ft above sea level.

3. A child who weighs 34 kg is seated on a seesaw, while a child who weighs 40kg is situated on the opposite end of the seesaw. The function $D(x) = \dfrac{34x}{40}$ gives the distance that the 40 kg child must sit from the center of the seesaw when the 34 kg child sits *x* meters from the center. The seesaw is 9m long. Find the average rate of change in distance as the lighter child's distance changes from 1.5m to 2.5m, and find the instantaneous rate of change at 2.5m.

4. The pitch ***p*** of a tone and its wavelength ***w*** in meters are related to the velocity ***v*** in m/s of sound through air and can be modeled by the function $p = \dfrac{v}{w}$. For a sound wave with velocity of 985m/s, find the average rate of change in pitch as the wavelength changes from 35m to 45m, and find the instantaneous rate of change at a wavelength of 45m.

5. The average speed of a certain particle in meters per second is given by the equation $S(t) = \dfrac{2t^2 + 8t + 5}{t + 3}$. Find the average rate of change as time changes from 0.5 seconds to 3.5 seconds, and find the instantaneous rate of change at 3.5 seconds.

6. After you eat something that contains sugar, the pH or acid level in your mouth changes. This can be modeled by the function $L(m) = \dfrac{-20.4m}{m^2 + 36} + 6.5$, where ***L*** is the pH level and ***m*** is the number of minutes that have elapsed since eating. Find the average rate of change from 1.5 minutes to 3 minutes, and find the instantaneous rate of change at 3 minutes.

2.5.1 Rate of Change Problems (Continued)

7. Airbags are a standard safety feature of most cars. Without airbags, a person or object would move forward at the velocity that the car is moving. A person inside a car traveling at 8m/s and weighing 8000kg would follow the function $v(m) = \frac{8000 - m}{8000 + m} \times 5$, where ***m*** represents the mass of the person. Find the average rate of change as the mass of the person changes from 57 kg to 75 kg, and the instantaneous rate of change at 75 kg.

8. The ratio of surface area to volume of a cylinder of radius 7cm is given by the function $f(h) = \frac{14\pi h + 98\pi}{49\pi h}$, where ***h*** represents the height of the cylinder in cms. Find the average rate of change as height changes from 4cm to 9cm, and find the instantaneous rate of change at 9cm.

9. Explain why the line ***y = x*** is an asymptote for the graph of $y = \frac{x^3}{x^2 + 1}$.

10. Explain why the line ***y = -x*** is an asymptote for the graph of $y = \frac{1 - x^2}{x}$.

ANSWERS:

Average Rate of Change	Instantaneous Rate of Change
1. 0.9629	1.1
2. -0.0065	-0.0069
3. -1.275	0.85
4. -0.6254	-0.4864
5. 2.3932	2
6. -0.3733333	-0.2720
7. -0.0012296	-0.0012
8. -0.055554	-0.02
9 & 10. Has an oblique asymptote because the degree of the numerator is exactly one greater than that of the denominator. The equation of that asymptote is found through synthetic division (the quotient).	