Unit 3 — Calculus and Vectors
Applying Properties of Derivatives

Lesson Outline

Day	Lesson Title	Math Learning Goals	Expectations
1, 2, 3,	*The Second Derivative* *(Sample Lessons Included)*	• Define the second derivative • Investigate using technology to connect the key properties of the second derivative to the first derivative and the original polynomial or rational function (increasing and decreasing intervals, local maximum and minimum, concavity and point of inflection) • Determine algebraically the equation of the second derivative $f''(x)$ of a polynomial or simple rational function $f(x)$, and make connections, through investigation using technology, between the key features of the graph of the function and those of the first and second derivatives	B1.1, B1.2, B1.3
4	*Curve Sketching from information*	• Describe key features of a polynomial function and sketch two or more possible graphs of a polynomial function given information from first and second derivatives – explain why multiple graphs are possible.	B1.4
5, 6	*Curve Sketching from an Equation*	• Extract information about a polynomial function from its equation, and from the first and second derivative to determine the key features of its graph • Organize the information about the key features to sketch the graph and use technology to verify.	B1.5
7	*Jazz Day*		
8	*Unit Summative* *(Sample Assessment Included)*		

Note: The assessment on day 8, and an assessment for a jazz day, is available from the member area of the OAME website and from the OMCA website (www.omca.ca).

Unit 3: Day 1: Concavity of Functions and Points of Inflection

MCV4U

Minds On: 15 Action: 45 Consolidate:15 Total=75 min	**Learning Goals**: • Recognize points of inflection as points on a graph of continuous functions where the concavity changes • Sketch the graph of a (generic)derivative function, given the graph of a function that is continuous over an interval, and recognize points of inflection of the given function (i.e., points at which the concavity changes)	**Materials** • Graphing Calculators • BLM3.1.1 • BLM3.1.2 • BLM3.1.3 • Computer lab or computer with data projector • Chart paper and markers
		Assessment Opportunities
Minds On…	**Pairs or Whole Class → Discussion** If enough computers are available, have students work in pairs with GSP to complete BLM 3.1.1 or, use the GSP Sketch **PtsOfInflection.gsp** as a demonstration and BLM 3.1.1 as a guide for discussion. *Key understanding*: "When the first derivative of a function "changes direction", there is a point of inflection, and the concavity changes:	Observe students working with GSP sketch. Listen to their use of function terminology.
Action!	**Small Groups→ Investigation** In heterogeneous groups of 3 or 4, students complete BLM 3.1.2 in order to investigate points of inflection for different polynomials. **Mathematical Process:** Connecting, Representing	
Consolidate Debrief	**Small Groups → Class Sharing** In heterogeneous groups of three, and using chart paper, students write down their observations from the investigations. Call on each group to share observations with the class. Groups post the chart paper with their observations from the investigations on the wall. Invite students to record information from the chart paper in their notebooks.	
Concept Practice *Reflection*	**Home Activity or Further Classroom Consolidation** Complete BLM 3.1.3 Journal Entry: "Do all graphs have points of inflection? Explain."	

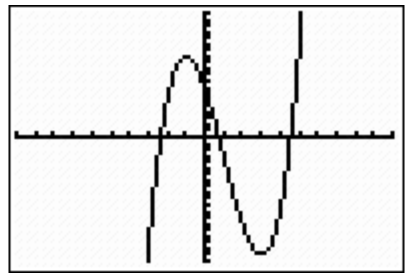
3.1.1 Concavity and Point(s) Of Inflection

1. For each of the following graphs of functions:

a) determine (approximately) the interval(s) where the function is concave up.
b) determine (approximately) the interval(s) where the function is concave down.
c) estimate the coordinates of any point(s) of inflection.

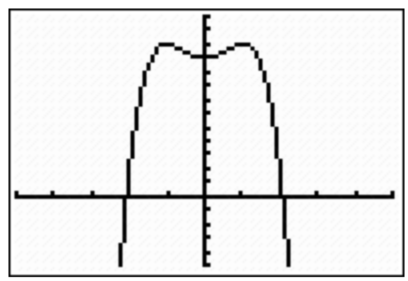
1.

2,



3.

4.



5.

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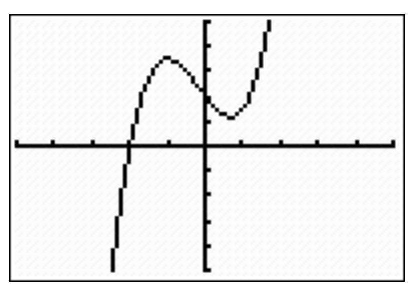
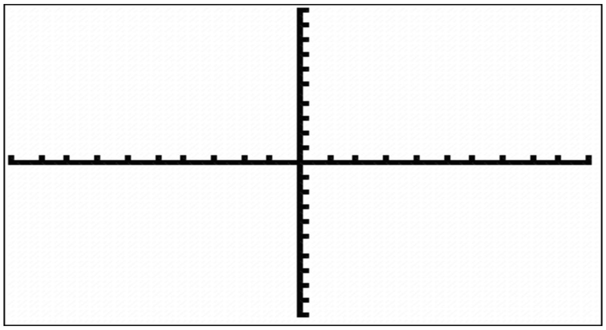


3.1.2 Investigate Points of Inflection

PART A

1. i) Using technology, graph $f(x)=x^3$, the first derivative $f'(x)=3x^2$, and the second derivative $f''(x)=6x$ on the same viewing screen.

ii) Copy and label each graph on the grid below.



iii) Identify the point of inflection of $f(x)=x^3$ using the TABLE feature of the graphing calculator.

iv) Show algebraically that the point where the first derivative, $f'(x)=3x^2$, of the function, $f(x)=x^3$, is equal to 0 is (0, 0).

v) For the interval $x\in(-\infty, 0)$, the first derivative $f'(x)=3x^2$ of the function $f(x)=x^3$ is above the *x*-axis and is ________________. For the interval $x\in(-\infty, 0)$, the function $f(x)=x^3$ is a(n) __________________ function.

For the interval $x\in(0,\infty)$, the first derivative $f'(x)=3x^2$ of the function $f(x)=x^3$ is also above the *x*-axis and is ________________. For the interval $x\in(0,\infty)$, the function $f(x)=x^3$ is a(n) ____________________ function.

vi) Show algebraically that the point where the second derivative, $f''(x)=6x$, of the function, $f(x)=x^3$, is equal to 0 is (0, 0).

vii) For the interval $x\in(-\infty, 0)$, the second derivative $f''(x)=6x$ of the function $f(x)=x^3$ is below the *x*-axis and is ________________. For the interval $x\in(-\infty, 0)$, the function $f(x)=x^3$ is concave ________.

For the interval $x\in(0,\infty)$, the second derivative $f''(x)=6x$ of the function $f(x)=x^3$ is above the *x*-axis and is ________________. For the interval $x\in(0,\infty)$, the function $f(x)=x^3$ is concave ________.

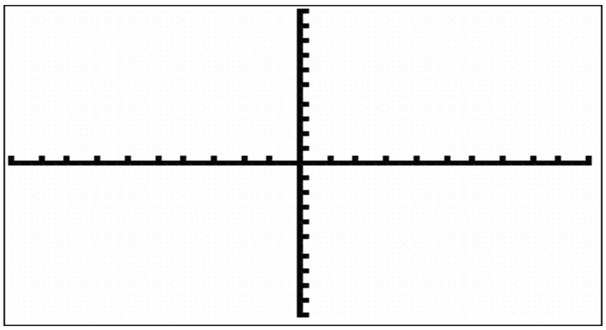
viii) Describe the behaviour of the second derivative of the function to the left and to the right of the point where the second derivative of the function $f(x)=x^3$ is equal to 0.

3.1.2 Investigate Points of Inflection (Continued)

PART B

1. i) The graph of $f(x)=x^3+2$ is the graph of $f(x)=x^3$ translated ______ units ________. Using technology, create the graphs of the function $f(x)=x^3$; and the function $f(x)=x^3+2$, the first derivative of the function $f'(x)=3x^2$, and the second derivative of the function $f''(x)=6x$ in the same viewing screen.

ii) Copy and label each graph on the grid below.



iii) Identify the point of inflection of $f(x)=x^3+2$.

iv) Show algebraically that the point where the first derivative, $f'(x)=3x^2$, of the function, $f(x)=x^3+2$, is equal to 0 is (0, 0).

v) For the interval $x\in(-\infty,0)$, the first derivative $f'(x)=3x^2$ of the function $f(x)=x^3+2$ is above the *x*-axis and is ________________. For the interval $x\in(-\infty,0)$, the function $f(x)=x^3+2$ is a(n) __________________ function.

For the interval $x\in(0,\infty)$, the first derivative $f'(x)=3x^2$ of the function $f(x)=x^3+2$ is also above the *x*-axis and is ________________. For the interval $x\in(0,\infty)$, the function $f(x)=x^3+2$ is a(n) __________________function.

vi) Show algebraically that the point where the second derivative, $f''(x)=6x$, of the function, $f(x)=x^3+2$, is equal to 0 is (0, 0).

vii) For the interval $x\in(-\infty,0)$, the second derivative $f''(x)=6x$ of the function $f(x)=x^3+2$ is below the *x*-axis and is ________________. For the interval $x\in(-\infty,0)$, the function $f(x)=x^3+2$ is concave ________.

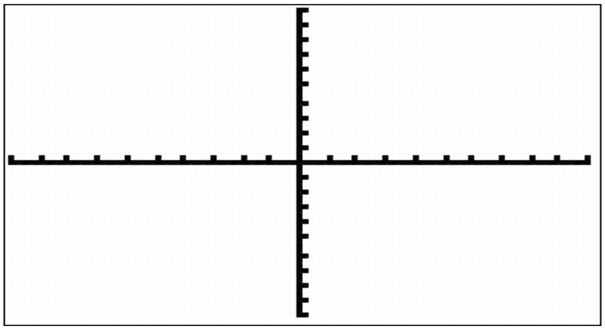
For the interval $x\in(0,\infty)$, the second derivative $f''(x)=6x$ of the function $f(x)=x^3+2$ is above the *x*-axis and is ________________. For the interval $x\in(0,\infty)$, the function $f(x)=x^3+2$ is concave ________.

viii) Describe the behaviour of the second derivative of the function to the left and to the right of the point where the second derivative of the function $f(x)=x^3+2$ is equal to 0. What effect does translating a function vertically upwards have on the location of a point of inflection for a given function?

3.1.3 Sketch the Graph Of the Derivative Of A Function

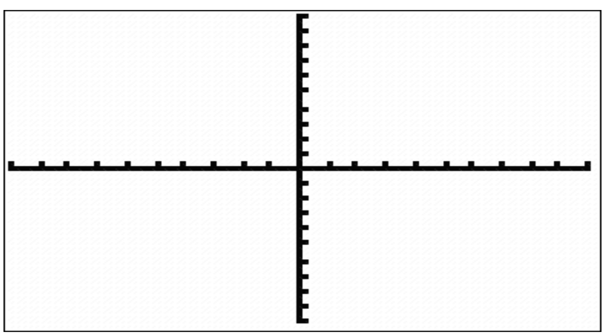
1. a) Determine the first and second derivatives of the function $f(x) = -x^3 + 3$.

b) Sketch the first and second derivatives of the function $f(x) = -x^3 + 3$, and use these graphs to sketch the graph of the function $f(x) = -x^3 + 3$ on the grid below.



2. a) Determine the first and second derivative of each of the function $f(x) = -(x+3)^3$.

b) Sketch the graphs of the first and second derivatives of the function $f(x) = -(x+3)^3$, and use these graphs to sketch the graph of the function $f(x) = -(x+3)^3$ on the grid below.



3. Explain how you can determine the point of inflection for a function by studying the behaviour of the second derivative of the function to the left and to the right of the point where the second derivative of the function is equal to zero.

Unit 3: Day 2: Determining Second Derivatives of Functions

MCV4U

Minds On: 15 Action: 50 Consolidate:15 Total=75 min	**Learning Goals**: • Define the second derivative of a function. • Determine algebraically the equation of the second derivative $f''(x)$ of a polynomial or simple rational function $f(x)$.	**Materials** • Graphing Calculators • BLM3.2.1 • BLM3.2.2 • BLM3.2.3 • BLM3.2.4 Large sheets of graph paper and markers

		Assessment Opportunities
Minds On…	**Pairs → Investigation** Students work in pairs to complete BLM 3.2.1 Establish the definition of the second derivative. The definition should include “rate of change” and “tangent line”..	
Action!	**Small Groups→ Guided Exploration** In heterogeneous groups of four students complete BLM 3.2.2 and BLM 3.2.3 . **Mathematical Process:** Selecting tools, Communicating	Use a checklist or checkbric to note whether students can apply the Product Rule.
Consolidate Debrief	**Individual → Journal** Students write a summary of findings from BLM 3.2.3 in their journal	
Application	**Home Activity or Further Classroom Consolidation** Use BLM 3.2.4 to consolidate knowledge of and practise finding first and second derivatives using differentiation.	

3.2.1 Find the Derivatives of Polynomial and Rational Functions

1. Find the derivative of each function. Determine the derivative of the derivative function.

a) $f(x) = x^2 + 4x + 3$

b) $f(x) = 3x^2 - 5x + 2$

c) $f(x) = \dfrac{x+1}{x-2}$

d) $f(x) = \dfrac{x^2}{x^2 - 1}$

3.2.2 The First and Second Derivatives of Polynomial and Rational Functions

1. Complete the following chart:

Function, $f(x)$	**First Derivative,** $f'(x)$	**Second Derivative,** $f''(x)$
a) $f(x) = 6$		
b) $f(x) = -3x + 2$		
c) $f(x) = 2x^2 + 5x - 7$		
d) $f(x) = (x-3)^3(x+1)$		
e) $f(x) = 4x^3 + 5x^2 + 7x + 3$		
f) $f(x) = -6x^3 + 3x^2 - 2x + 5$		
g) $f(x) = 2x^4 - 4x^3 + 6x^2 - 3x + 2$		

3.2.2 The First and Second Derivatives of Polynomial and Rational Functions (Continued)

2. Complete the following chart: [LEAVE YOUR ANSWERS IN UNSIMPLIFIED FORM.]

Function, $f(x)$	First Derivative, $f'(x)$	Second Derivative, $f''(x)$
a) $f(x)=\dfrac{x+1}{x-2}$		
b) $f(x)=\dfrac{x}{x^2-4}$		
c) $f(x)=\dfrac{x^2}{x^2-9}$		
d) $f(x)=\dfrac{x^2-1}{(x+1)^5}$		

3.2.3 Working Backwards from Derivatives

1. The first derivative *f'(x)* is given. Determine a possible function, *f(x)*.

First derivative, $f'(x)$	**Possible function,** $f(x)$ **?**
a) $f'(x) = 2x + 4$	$f(x) = x^2 + 4x + 1$ ← **Note**: Any constant could be added or subtracted here.
b) $f'(x) = 7$	
c) $f'(x) = 2(x+1)$	
d) $f'(x) = 3x^2 + 2x + 6$	
e) $f'(x) = -(x+1)^{-2}$	

3.2.4 Home Activity: First and Second Derivatives

1. Complete the following chart.

Function, $f(x)$	**First Derivative,** $f'(x)$	**Second Derivative,** $f''(x)$
a) $f(x) = 3x^2 + 7x - 5$		
b) $f(x) = (2x+3)^2$		
c) $f(x) = 3x^4 - 5x^3 + 4x^2 - 8x + 3$		
d) $f(x) = \dfrac{x+3}{x-5}$		
e) $f(x) = \dfrac{x}{x^2 - 16}$		

2. Create a polynomial function of degree 2. Assume that your function (*f'(x)*) is the derivative of another function (*f(x)*). Determine a possible function (*f(x)*).

Unit 3: Day 3: The Second Derivative, Concavity and Points of Inflection

MCV4U

Minds On: 10 Action: 50 Consolidate:15 Total=75 min	**Learning Goals**: • Determine algebraically the equation of the second derivative $f''(x)$ of a polynomial or simple rational function $f(x)$, and make connections, through investigation using technology, between the key features of the graph of the function and those of the first and second derivatives	**Materials** • graphing calculators • Computer and data projector • BLM 3.3.1 • BLM 3.3.2 • BLM 3.3.3 • BLM 3.3.4

		Assessment Opportunities
Minds On…	**Pairs → Pair Share** Students coach each other as they complete a problem similar to the work from the previous class. (**A** coaches B, and B writes, then reverse roles) **Whole Class → Discussion** Using the GSP sketch **PtsOfInflection.gsp,** review points of inflection, and the properties of the first and second derivatives with respect to concavity and points of inflection	Each pair of students has only one piece of paper and one writing instrument.
Action!	**Pairs → Guided Exploration** **Curriculum Expectation/Observation/Mental note:** Circulate, listen and observe for student's understanding of this concept as they complete BLM 3.3.1 and 3.3.2. Students work in pairs to complete the investigation on BLM3.3.1, BLM3.3.2. As students engage in the investigation, circulate to clarify and assist as they: • determine the value(s) of the first and second derivatives of functions for particular intervals and x-values. • use a graphing calculator to determine the local maximum. and/or minimum point(s) and point(s) of inflection of a polynomial function. • understand the connection between the sign of the second derivative and the concavity of a polynomial function • use the second derivative test to determine if a point if a local maximum. point or a local minimum point for a polynomial functions. **Mathematical Process: R**eflecting, representing	
Consolidate Debrief	**Whole Class → Discussion** Lead a discussion about the information provided by the second derivative. Students should be able to articulate their understanding about the properties of the second derivative.	
Exploration *Application*	**Home Activity or Further Classroom Consolidation** Complete BLM3.3.3 in order to consolidate your understanding of first derivatives, second derivatives, local max./minimum points and points of infection for a simple rational function. Complete a Frayer Model for the "Second Derivative" on BLM 3.3.4.	See pages 22-25 of THINK LITERACY : Cross - Curricular Approaches , Grades 7 - 12 for more information on Frayer Models.

3.3.1 The Properties of First and Second Derivatives

Investigate:

1. a) Graph each of the following functions on the same viewing screen of a graphing calculator.

$$f(x) = x^2 - 2x - 2$$

$$f'(x) = 2x - 2$$

$$f''(x) = 2$$

b) Classify each of the functions in part **a)** as a quadratic function, a linear function, or a constant function.

$f(x) = x^2 - 2x - 2$ ______________________ function

$f'(x) = 2x - 2$ ______________________ function

$f''(x) = 2$ ______________________ function

c) How are the first and second derivative functions related to the original function?

d) Determine the coordinates of the vertex of the function $f(x) = x^2 - 2x - 2$.

e) Determine if the vertex of the function $f(x) = x^2 - 2x - 2$ is a local maximum or a local minimum point.

f) Does the function $f(x) = x^2 - 2x - 2$ have a point of inflection? Explain.

g) Use a graphing calculator to determine the value of the first derivative of the function $f(x) = x^2 - 2x - 2$ for each *x*-value.

$x = -1$ $f'(-1) =$

$x = 1$ $f'(1) =$

$x = 3$ $f'(3) =$

3.3.1 The Properties of First and Second Derivatives (continued)

h) Complete the following chart by circling the correct response for each interval.

Interval	$x \in (-\infty, 1)$	$x = 1$	$x \in (1, \infty)$
The first derivative $f'(x) = 2x - 2$	- is positive - is zero - is negative	- is positive - is zero - is negative	- is positive - is zero - is negative
The function, $f(x) = x^2 - 2x - 2$	- is increasing - has a local max. - has a local min. - is decreasing	- is increasing - has a local max. - has a local min. - is decreasing	- is increasing - has a local max. - has a local min. - is decreasing
The second derivative $f''(x) = 2$ is	- is positive - is zero - is negative	- is positive - is zero - is negative	- is positive - is zero - is negative
The function, $f(x) = x^2 - 2x - 2$	- is concave up - has a pt. of inflection - is concave down	- is concave up - has a pt. of inflection - is concave down	- is concave up - has a pt. of inflection - is concave down

The first derivative, $f'(x) = 2x - 2$, of the function $f(x) = x^2 - 2x - 2$ is _______________ when x = 1 and the second derivative, $f''(x) = 2$, of the function $f(x) = x^2 - 2x - 2$ is ____________ when x = 1. A local _______________ point occurs when x = 1.

The second derivative, $f''(x) = 2$, of the function $f(x) = x^2 - 2x - 2$ is ____________ for all values of x. The function $f(x) = x^2 - 2x - 2$ is concave __________ for all values of x and does _____ have a point of inflection.

Consolidate:

If the second derivative of a function is positive when the first derivative of a function is equal to

zero for a particular x-value, then a local ________________ point will occur for that particular x-value.

If the second derivative of a function is negative when the first derivative of a function is equal to

zero for a particular x-value, then a local ________________ point will occur for that particular x-value.

3.3.2 Derivatives of a Cubic Function

Investigate:

1. a) Graph each function on the same viewing screen of a graphing calculator.

$$f(x) = x^3 - 3x^2 - 9x + 15$$

$$f'(x) = 3x^2 - 6x - 9$$

$$f''(x) = 6x - 6$$

Use the following WINDOW settings:

Xmin = −10
Xmax = 10
Xscl = 1
Ymin = −20
Ymax = 20
Yscl = 1
Xres = 1

b) Identify each of the functions in part a) as a cubic function, a quadratic function, a linear function, or a constant function.

$f(x) = x^3 - 3x^2 - 9x + 15$ ________________________ function

$f'(x) = 3x^2 - 6x - 9$ ________________________ function

$f''(x) = 6x - 6$ ________________________ function

c) How are the first and second derivative functions related to the original function?

d) Write the coordinates of the local maximum point of the function $f(x) = x^3 - 3x^2 - 9x + 15$.

e) Write the coordinates of the local minimum point of the function $f(x) = x^3 - 3x^2 - 9x + 15$.

f) Write the coordinates of the point of inflection for the function $f(x) = x^3 - 3x^2 - 9x + 15$.

3.3.2 Derivatives of a Cubic Function (Continued)

g) Complete the following chart by circling the correct response for each interval.

Interval	$-\infty < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < \infty$
The first derivative $f'(x) = 3x^2 - 6x - 9$	positive zero negative	positive zero negative	Positive zero negative	positive zero negative	positive zero negative	positive zero negative	positive zero negative
The function, $f(x) = x^3 - 3x^2 - 9x + 15$	increasing . decreasing	local max. local min.	increasing . decreasing	local max. local min.	increasing . decreasing	local max. local min.	increasing . decreasing
The second derivative $f''(x) = 6x - 6$	positive zero negative	positive zero negative	positive zero negative	positive zero negative	positive zero negative	positive zero negative	positive zero negative
The function, $f(x) = x^3 - 3x^2 - 9x + 15$	concave up concave down	concave up pt. of inflection concave down	concave up concave down	concave up pt. of inflection concave down	concave up concave down	concave up pt. of inflection concave down	concave up concave down

Summarize the information in the table in your own words..

Consolidate:

If the second derivative of a function is zero for a particular *x*-value on the curve and has the opposite sign for points on either side of that particular *x*-value, then a ___________ of the function and a _______________ of the first derivative of the function will occur for that particular *x*-value.

3.3.3 Derivatives of Rational Functions

Investigate:

1. a) Graph each of the following functions on the same viewing screen of a graphing calculator. Use the standard viewing window.

$f(x) = \dfrac{x}{x^2 - 1}$	$f'(x) = \dfrac{-x^2 - 1}{(x^2 - 1)^2}$	$f''(x) = \dfrac{2x^3 + 6x}{(x^2 - 1)^3}$

b) Write the coordinates of the point of inflection for the function. $f(x) = \dfrac{x}{x^2 - 1}$

c) Complete the following chart by circling the correct response for each interval.

Interval	$-\infty < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < \infty$
The first derivative $f'(x) = \dfrac{-x^2 - 1}{(x^2 - 1)^2}$	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist
The function, $f(x) = \dfrac{x}{x^2 - 1}$	increasing . decreasing	local max. local min. not continuous	increasing . decreasing	local max. local min. not continuous	increasing . decreasing	local max. local min. not continuous	increasing . decreasing
The second derivative $f''(x) = \dfrac{2x^3 + 6x}{(x^2 - 1)^3}$	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist	positive zero negative does not exist
The function, $f(x) = \dfrac{x}{x^2 - 1}$	concave up concave down	concave up pt. of inflection concave down not continuous	concave up concave down	concave up pt. of inflection concave down not continuous	concave up concave down	concave up pt. of inflection concave down not continuous	concave up concave down

d) Explain the behaviour of the function $f(x) = \dfrac{x}{x^2 - 1}$ at the points where $x = -1$ and $x = 1$.

3.3.4 Home Activity: Frayer Model

Name______________________________ Date___________________________

Definition	Facts/Characteristics
Examples	Non-examples

Second Derivative of a Function