MCV4U
Calculus and Vectors
University Preparation
Unit 1

To be updated the week of August 27 and further edited in September 2007

Unit 1: Rates of Change

Grade 12

Lesson Outline

Big Picture

Students will:

- connect slopes of secants to average rates of change, and slopes of tangents to instantaneous rates of change in a variety of contexts.
- approximate rates of change graphically and numerically.

Day	Lesson Title	Math Learning Goals	Expectations
1	Rates of Change Revisited *(Sample Lesson Included)*	• Describe real-world applications of rate of change (e.g., flow) problems using verbal and graphical representations (e.g., business, heating, cooling, motion, currents, water pressure, population, environment, transportations) • Describe connections between average rate of change and slope of secant, and instantaneous rate of change and slope of tangent in context	A1.1, A1.2
2	Determine Instantaneous Rate of Change using Technology *(Sample Lesson Included)*	• With or without technology, determine approximations of and make connections between instantaneous rates of change as secant lines tends to the tangent line in context	A1.3
3	Exploring the Concept of a Limit *(Sample Lesson Included)*	• Explore the concept of a limit by investigating numerical and graphical examples and explain the reasoning involved • Explore the ratio of successive terms of sequences and series (use both divergent and convergent examples) • (e.g., Explore the nature of a function that approaches an asymptote (horizontal and vertical)	A1.4
4–5	Calculating an instantaneous rate of change using a numerical approach *(Sample Lesson Included)*	• Connect average rate of change to $\frac{f(a+h)-f(a)}{h}$ and instantaneous rate of change to $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$	A1.5, A1.6
6–7	Jazz/ Summative	•	

Unit 1: Day 1: Revisiting Rates of Change — Grade 12

75 min

Math Learning Goals

- Describe real-world applications of rate of change using verbal and graphical representations (e.g., business, heating, cooling, motion, currents, water pressure, population, environment, transportation).
- Describe connections between average rate of change and instantaneous rate of change in context.

Materials

- chart paper and markers
- computer and data projector
- BLMs 1.1.1 – 1.1.5

Assessment Opportunities

Minds On…

Groups → Graffiti

Prepare and post 7 pieces of chart paper each containing a term students encountered in MHF4U. Give each group a different coloured marker.

In heterogeneous groups of 3 or 4 students have 30 seconds to write anything they know about the term using numbers, symbols and/or words. Groups move through all seven pieces of chart paper.

Whole Class → Discussion

Using the four scenarios provided on BLM 1.1.1, review connections between rates of change and the slopes of secants and tangents. Guiding questions:

- Describe the rate of change of the walleye population over the 25 year period.
- Would you expect half of the water to drain in half the time? Justify.
- What is the rocket's instantaneous rate of change at 4 seconds? Describe what the rocket is doing at this point of time.

Although the Ferris Wheel is turning at a constant rate, the rate of change of height is not constant. Explain why.

Graffiti Terms:
- Dependent variable
- Independent variable
- Finite differences
- Slope of secant line
- Slope of tangent line
- Average rate of change
- Instantaneous rate of change

See pp. 26–28 of *Think Literacy: Cross-Curricular Approaches, Grades 7–12* for more information on Graffiti.

Action!

Pairs → Investigation

Curriculum Expectation A1.2/Observation/Mental Note: Observe to identify students ability to make connections between the average rate of change and slope of secant and instantaneous rate of change and slope of tangent.

Students complete the investigation on average and instantaneous rates of change from BLM 1.1.2.

Consider using a computer lab with GSP® to complete the investigation on BLM 1.1.2.

Mathematical Process Focus: Connecting

GSP® sketch Ball Bounces.gsp can be used to demonstrate. Make use of a smartboard if available.

See pp. 22–25 of *Think Literacy: Cross-Curricular Approaches, Grades 7–12* for more information on Frayer Models

Consolidate Debrief

Whole Group → Discussion

Share findings with the class. Address any misunderstandings.

Guiding Questions: Describe how to select points on a curve so that the slope secant better represents the instantaneous rate of change at any point in the interval?

How would you change the intervals around each bounce to provide better information about the average and instantaneous rates of change of the ball?

Pairs → Pair/Share: Frayer Model

A coaches B in completing Frayer model from BLM 1.1.4 for average rate of change. B coaches A in completing Frayer model from BLM 1.1.4 for instantaneous rate of change.

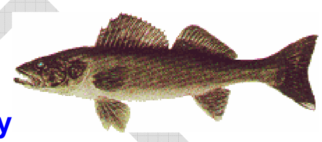
Application

Home Activity or Further Classroom Consolidation

Students gather examples of rates of change from their own lives using BLM 1.1.5.

1.1.1: Revisiting Rates of Change

25-year Walleye Population	
Year	**Walleye Population**
0	3000
1	3400
2	3720
3	3976
4	4181
5	4345
6	4476
7	4581
8	4665
9	4732
10	4786
11	4829
12	4863
13	4890
14	4912
15	4930
16	4944
17	4955
18	4964
19	4971
20	4977
21	4982
22	4986
23	4989
24	4991
25	4993

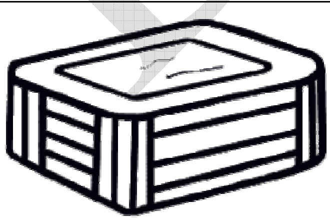


A fish story

A pond was stocked with a type of fish called a "walleye." The table on the left gives the population of walleye in the pond for the 25 years following the stocking of the pond.

Down the Drain

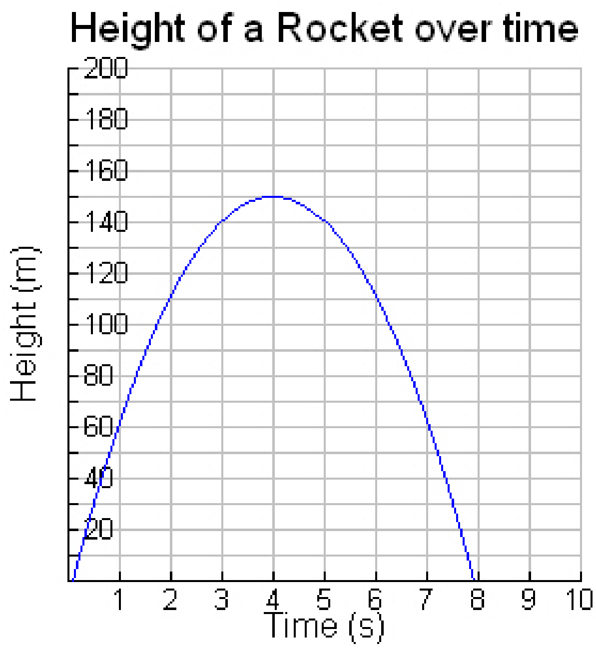
The plug is pulled in a small hot tub. The table on the right gives the volume of water in the tub from the moment the plug is pulled, until it is empty.

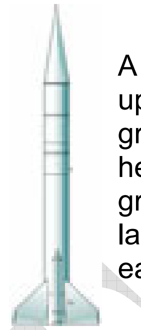


Source:
http://www.clipsahoy.com/webgraphics/as0963.htm

Draining Water from a Hot Tub	
Time (s)	**Volume (L)**
0	1600
10	1344
20	1111
30	900
40	711
50	544
60	400
70	278
80	178
90	100
100	44
110	11
120	0

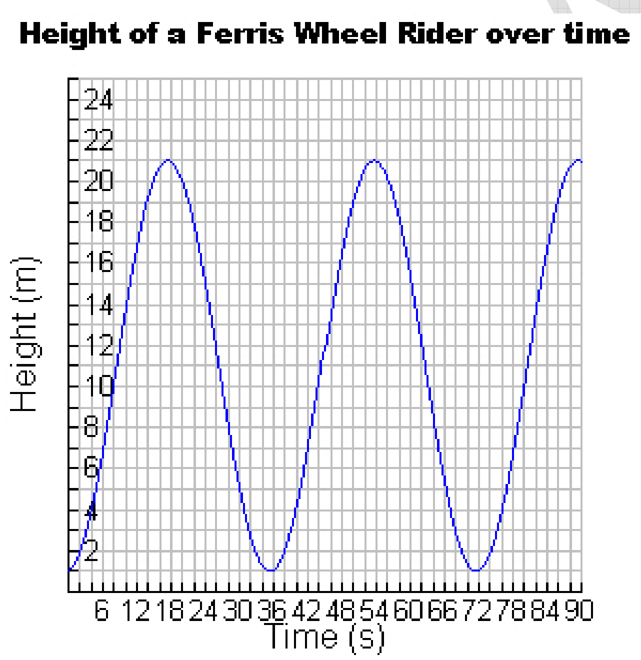
1.1.1: Revisiting Rates of Change (continued)





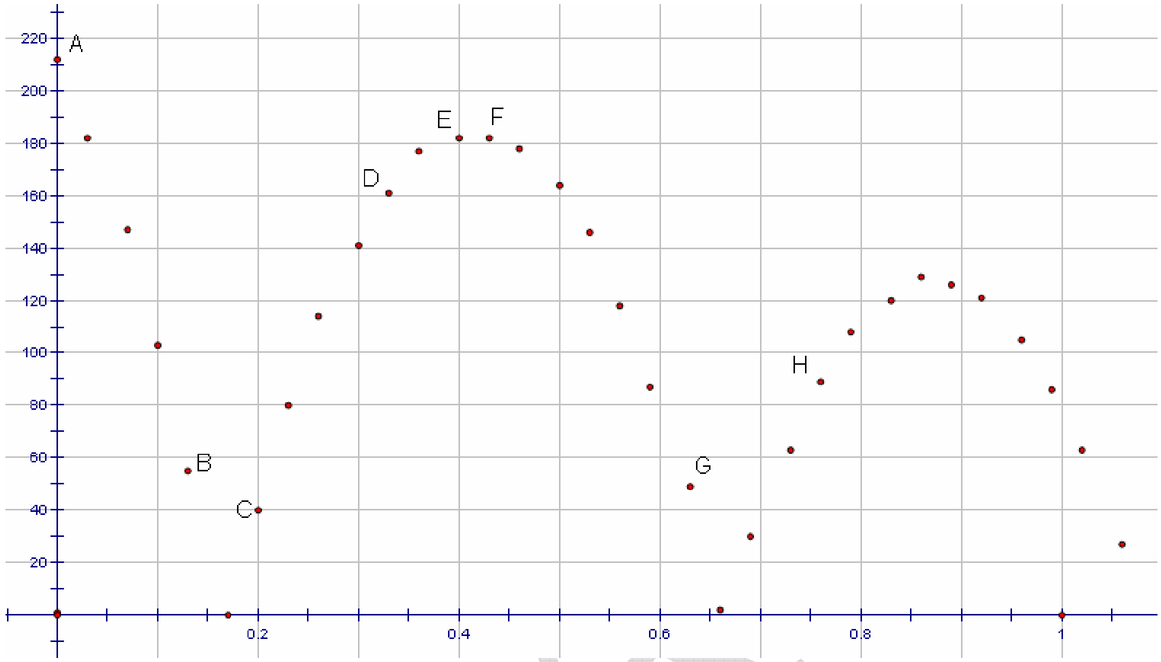
Blast Off

A rocket is launched up in the air. The graph shows it's height above the ground from time of launching to return to earth.



1.1.2: That's the Way the Ball Bounces!

Kevin dropped a ball and he collected the height (m) at various times (s). A graph of the data he collected is provided.



Part A: Average Rate of Change

Kevin wants to look at rate of change of the height at various times. He is hoping to determine how quickly the height was changing at various times. Kevin first wants to look at the average rate of change for specific time intervals. Complete the table for him with the information in the graph.

Interval	Coordinates of End Points		Average Rate of Change
AB			
BC			
CD			
DE			
EF			
FG			
GH			

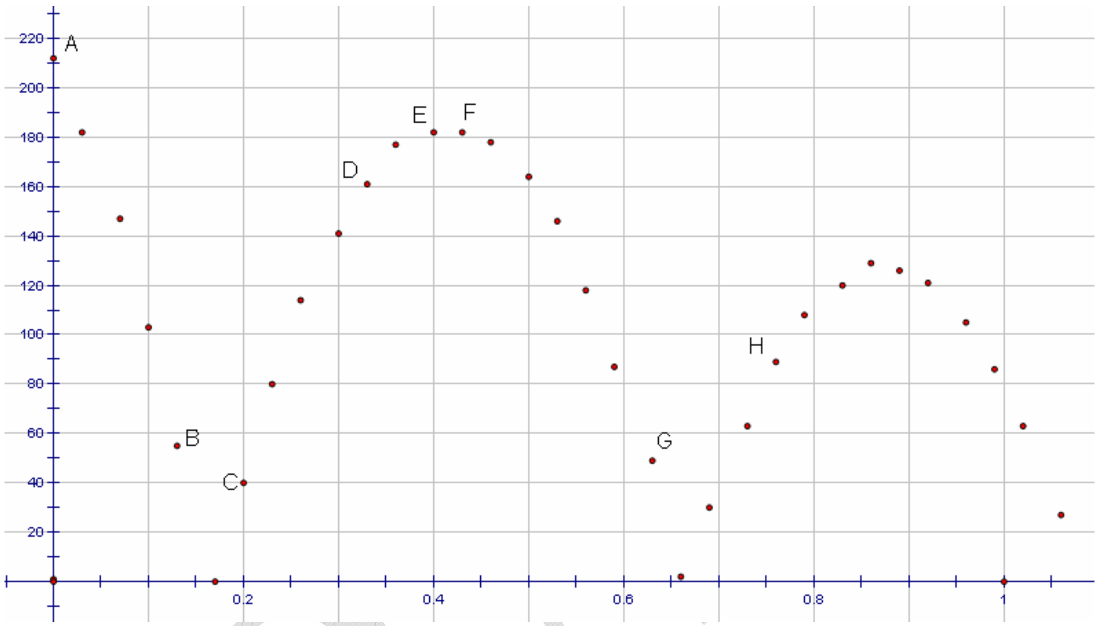
What do the values for average rate of change tell you about the path of the ball and the speed of the ball?

1.1.2: That's The Way the Ball Bounces! (Teacher)

Part B: Instantaneous Rate of Change

Next, Kevin wants to find a point in each interval whose tangent has the same instantaneous rate of change as each secant.

Sketch a curve of best fit for this data in one colour and draw the secants for the intervals in another colour.



For each time interval, locate the point on the graph between the endpoints whose tangent appears to have the same rate of change as the average rate of change for that interval.

1) On the interval AB the average rate of change is ____________. The point whose tangent matches this rate of change is (_________, _________). Draw the instantaneous rate of change at this point if possible.

2) On the interval BC, the average rate of change is ____________. The point that most closely matches this rate of change is (_________, _________).

3) On the interval CD, the average rate of change is ____________. The point that most closely matches this rate of change is (_________, _________). Draw the tangent at this point if possible.

4) On the interval EF, the average rate of change is ____________. The point that most closely matches this rate of change is (_________, _________). Draw the tangent at this point if possible.

1.2.2: That's The Way the Ball Bounces! (continued)

5) On the interval GH, the average rate of change is ____________. The point that most closely matches this rate of change is (______, ______).

   Kevin notices some problems for some of the intervals.

   a) For which intervals is it difficult to find a matching point?

   b) Why is if difficult to find instantaneous rate of change for these intervals?

   c) What was happening to the motion of the ball in these intervals?

1.1.3: That's The Way the Ball Bounces! (Teacher)

The following provides the coordinates of all the data points gathered by the ball bounce experiment.

Time (s)	Height (cm)
0.00	212
0.03	182
0.07	147
0.10	103
0.13	55
0.17	0
0.20	40
0.23	80
0.26	114
0.30	141
0.33	161
0.36	177
0.40	182
0.43	182
0.46	178
0.50	164
0.53	146
0.56	118
0.59	87
0.63	49
0.66	2
0.69	30
0.73	63
0.76	89
0.79	108
0.83	120
0.86	129
0.89	126
0.92	121
0.96	105
0.99	86
1.02	63
1.06	27

1.1.4: Frayer Model

Name: ______________________________ **Date:** ____________________

Definition	**Characteristics**
Examples	**Non-Examples**

Average Rate of Change

1.1.4: Frayer Model (continued)

Definition	Characteristics
Examples	**Non-Examples**

Instantaneous Rate of Change

1.1.4: Frayer Model Solutions (Teacher)

Definition	Characteristics
Average Rate of Change is the measure of the rate of change for a continuous function over a time interval.	• The rate can be represented as the slope of a secant line between the end points of the interval • The slope of the secant line is equivalent to the average rate of change • Cannot be determined over intervals for functions that have non-continuous intervals such as cusps and vertical asymptotes • The sign of the slope indicates whether a function is increasing or decreasing
Examples	**Non-Examples**
Average speed of a car for a trip Speed = $\frac{\text{distance travelled}}{\text{elapsed time}}$ Average rate of bacteria growth Slope of Secant $= \dfrac{f(a+h)-f(a)}{h}$	• Average height • Average class mark • Average income of families

Average Rate of Change

1.1.4: Frayer Model Solutions (Teacher)

Instantaneous Rate of Change

Definition

Instantaneous Rate of Change is the measure of the rate of change for a continuous function at point on the function.

Characteristics

- The rate can be represented as the slope of the tangent line to a curve at a particular point
- The slope of the tangent line is equivalent to the instantaneous rate of change
- Cannot be determined when there is a drastic change in the motion of an object such as at the point an object bounces
- Cannot be determined for functions that are not continuous or have vertical asymptotes

Examples

Real-time readout of speed of a car.
Real-time readout of a Geiger counter measuring radioactivity
Slope of Tangent to a curve

Non-Examples

- Average rate of change of a function
- Gauges that do not measure rates such as: odometer in a car, altimeter in an aircraft, …

1.1.5: Bringing It All Together: <u>Home Activity</u>

Describe an example in your life, your home, your vehicle, your sport, your _____? that matches each of the following situations. Explain why you believe each situation models the requirements stated.

Use any of the examples or situations different from those discussed in class today!

1. Positive average rate of change all of the time.

2. Positive average rate of change sometimes and a negative average rate of change sometimes.

3. Instantaneous rate of change equal to zero at least once.

4. Instantaneous rate of change which cannot be calculated at least once.

Unit 1: Day 2: Go with the Flow — Grade 12

Math Learning Goals
- Make connections with or without graphing technology between an approximate value of the instantaneous rate of change at a given point on the graph of a smooth function and average rates of change over intervals containing the point.
- Use the slopes of a series of secants through a given point on a smooth curve to approximate the slope of the tangent at the point.

75 min

Materials
- BLMs 1.2.1 – 1.2.6
- plastic bottles
- 1-2L graduated cylinders or measuring cups
- water
- stop watch

Assessment Opportunities

Minds On…

Pairs → Think/Pair/Share

Use the context of water flowing out of a water reservoir tank and BLMs 1.2.1 and 1.2.2 to activate prior knowledge about average and instantaneous rate of change and to help teachers assess what students know about rate of change.

Action!

Small Groups → Investigation

Curriculum Expectation A1.3/Observation/Mental Note: Observe to identify students' ability to make connections between the average rate of change over an interval containing a point and the instantaneous rate of change at a given point.

Students work in groups of three or four using BLM 1.2.3 to help understand the instantaneous rate of change of the volume of water as it flows out of a container with respect to time and to recognize this to be the instantaneous rate of flow. Students approximate the instantaneous rate of water flowing from a plastic drink container into a measuring cup or graduated cylinder using a series of secants to the graph showing the relationship between the volume of water flowing out of the plastic drink container and time. Hint Cards (BLM 1.2.4) are available to provide additional scaffolding for groups that need it.

BLM 1.2.5 provides instructions for using technology to graph the data and find slopes of secants.

Mathematical Process Focus: Connecting, Selecting Tools and Strategies.

Collect and prepare plastic cylindrical containers of various sizes.

DI Opportunity: The investigation can be changed to "height versus time" by placing a measured tape on the straight side of the container and adjusting the BLM appropriately.

Consolidate Debrief

Whole Class → Discussion

Students share results and strategies used to answer the questions in the investigation (BLM 1.2.3) with the whole class. Teacher uses the points made by students to consolidate the following:
- The difference between average and instantaneous rate of water flow.
- The connection between average rate of water flow with secants and instantaneous rate of water flow with tangents to the graph.
- The challenge of determining the instantaneous rate of water flow.
- The use of secants (or average rates of water flow) to approximate the instantaneous rate of water flow at a given point in time.

Concept Practice

Home Activity or Further Classroom Consolidation

Students use BLM 1.2.6 to consolidate the techniques developed in the activity.

1.2.1: Think/Pair/Share

Image source:
home.att.net/~berliner-Ultrasonics/bwzsagAa6.html

Context:
Answer each of the following regarding how water will flow out of the water reservoir shown. Assume the tank is cylindrical and the water is draining out of the bottom of the tank.

1. Do you think the rate of flow of the water out of the reservoir tank is constant? **Explain** your reasoning.

2. Using BLM 1.2.2, consider possible models for the relationship between the volume of water flowing out of the reservoir tank and time.
   Which model(s) would you immediately dismiss and why?

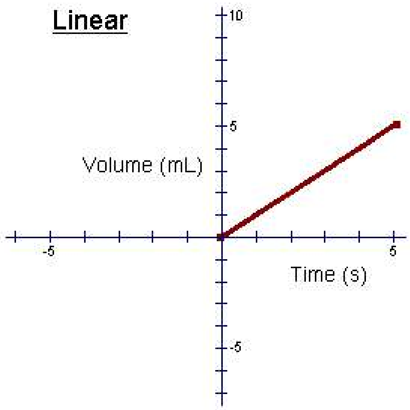
   Which graph best models the relationship between the total water that has flowed out of the tank and time? Justify your choice.

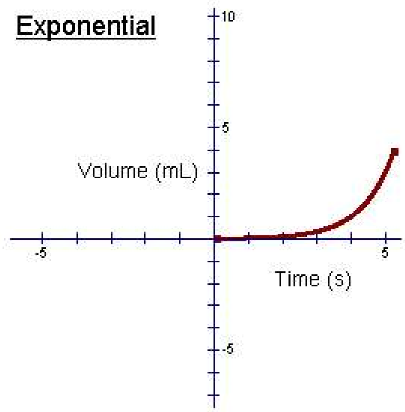
3. The average rate of flow is a measure of the rate of change of the volume of water that has flowed out of the reservoir over a given time interval. Use the graph chosen above to discuss how the average rate of flow changes as the reservoir empties.

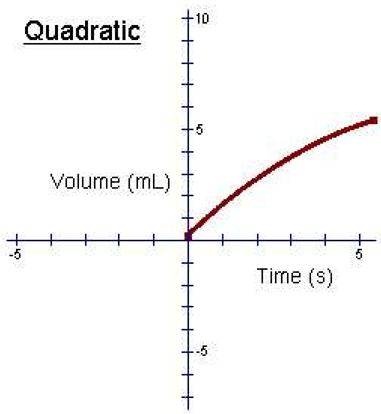
4. Describe the difference between an average rate of flow and an instantaneous rate of flow.

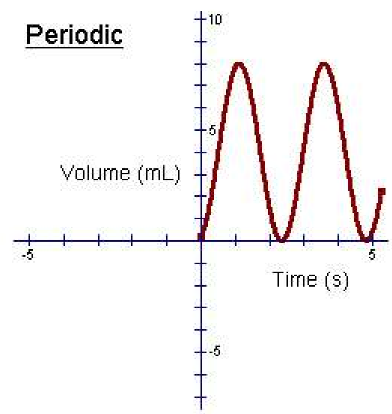
1.2.2: Average and Instantaneous Rate of Change

Each graph below models a relationship between the total volume (mL) of water that has flowed out of the tank and time (s).

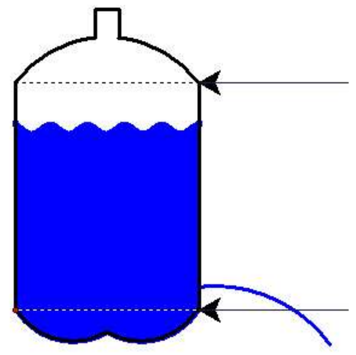








1.2.3: Go With the Flow



Context: To investigate the rate that water flows out of a cylindrical water tower.

Preparing Materials: Using a plastic drinking container (1.5–2L) make a 3–5 mm hole above the ridges on the bottom of the plastic drinking container. Set up the apparatus as shown in the diagram:

Step One: Gather volume and time data as one litre of water flows out of the plastic drinking container into a measuring cup or graduated cylinder. Record the time at which the volume in the measuring cup reaches a multiple of 50mL in the chart below:

Time (s)	Volume (mL)
0	0
	50
	100
	150
	200
	250
	300
	350
	400
	450
	500
	550
	600
	650
	700
	750
	800
	850
	900
	950
	1000

Graph the data, Volume versus time, using graph paper or graphing technology (e.g., graphing calculators, Excel© or Fathom™).

Step Two: Construct a curve of best fit with or without technology.

Step Three: Calculate the average rate of water flow over the whole time interval. (Hint Card 1) What connections can you make between the average rate of water flow over the whole time interval and the secant to the graph at the endpoints? (Hint Card 2)

Step Four: Repeat step 3 choosing two different points on the curve.

What connections can you make between instantaneous flow rates at a specific time and tangents to the graph? (Hint Card 3).
Approximate the instantaneous rate of flow, when 750 mL of water has been collected in the measuring cup, by using your graph and a series of secants containing the point. (Hint Card 4)

Step Five: Specifications for the water tower require that the rate of flow cannot be less than half the initial instantaneous flow rate.
What is the initial instantaneous rate of flow?
Investigate whether or not the rate of flow will meet the required specifications:

- when the container is half full.
- when the container is a quarter full.

Determine the time when the flow rate is exactly half the initial instantaneous flow rate.

1.2.4: Hint Cards

Hint 1:

$$Average\ Flow\ Rate = \frac{Volume\ Collected}{Time\ Interval}$$

If 200mL flows out of the container in 10 seconds, the average rate that the water flows out of the container is:

$$Average\ Flow\ Rate = \frac{Volume\ Collected}{Time\ Interval} = \frac{200mL}{10s} = 20\frac{mL}{s}$$

Hint 2:
A secant is a line that intersects a curve at two points. Find the slope of the secant using the first and last points on the graph. Compare with the average flow rate.

Hint 3:
A tangent is a line that makes contact with a curve at one point, without intersecting it. Find the slope of the tangent using two points on the line. Compare with the instantaneous flow rate.

Hint 4:
To use secants to approximate the slope of a given tangent find the slopes of secants with end points on either side of the point of tangency. To get better estimates of the slope of the tangent, make the end points of the secant closer to the point of tangency.

1.2.5: Go With the Flow: Average and Instantaneous Rate of Change (Teacher)

Analysis of Data using Fathom 2

1. Open the data collection: Go with the flow.ftm.
2. Complete **Step 1** by clicking on the table and entering each time and volume measurement. Note that because the volume collected is definitely 0 mL at time 0 s, record this data point first. If you are missing some volume-time measurements simply skip them or delete them from the table by right clicking the case number and selecting “Delete case.”
3. Complete **Step 2** by dragging the sliders for a, h and k (a is the vertical stretch factor, h is the horizontal shift constant, and k is the vertical shift constant.)
4. Complete **Step 3** by dragging the sliders for the time coordinates time_1 and time_2. Adjust the values to draw any secant of your choice. Note the slope of the secant (in the box at the bottom right hand corner of your screen.)

Analysis of Data using Geometer’s Sketchpad

1. Open the sketch: Go with the flow.gsp.
2. Complete **Step 1** by using the Graph menu and selecting Plot Points… Enter the time values as the x-coordinates and the volume values as the y-coordinates. Note that because the volume collected is definitely 0 mL at time 0 s, plot this data point first.
3. Complete **Step 2** by dragging the sliders for a, h and k (a is the vertical stretch factor, h is the horizontal shift constant, and k is the vertical shift constant.)
4. Complete **Step 3** by dragging point A and point B. Adjust the values to draw any secant of your choice. Note the slope of the secant.

Analysis of Data using Microsoft Excel

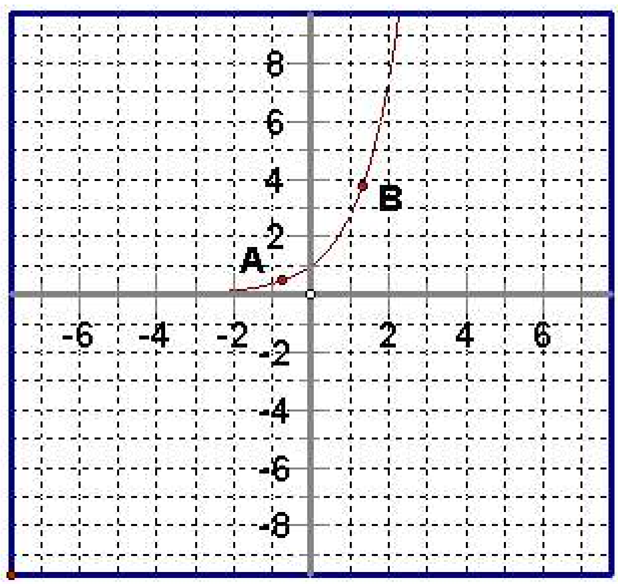
1. Open the file: Go with the flow.xls.
2. Complete **Step 1** by entering the time values with their corresponding volume values. Note that because the volume collected is definitely 0 mL at time 0 s, this is the first data point.
3. Note that the quadratic curve of best fit has been drawn for you and the equation for this curve is indicated in the bottom right.
4. Print the graph and draw secants as required. Determine the slope of the secant(s) by hand using the graph or the equation to determine the y-values of the points.

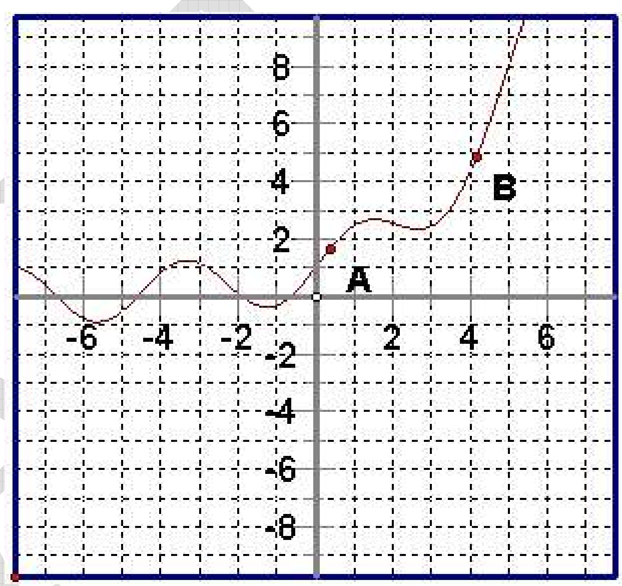
1.2.6: Go With the Flow

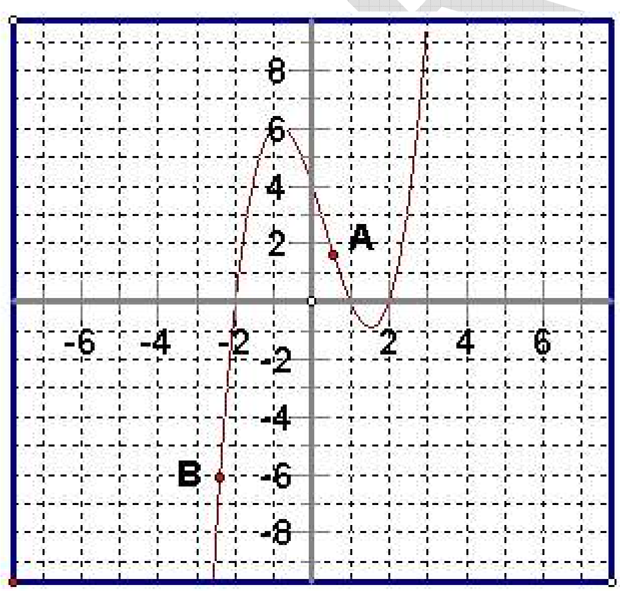
Follow Up Activity:

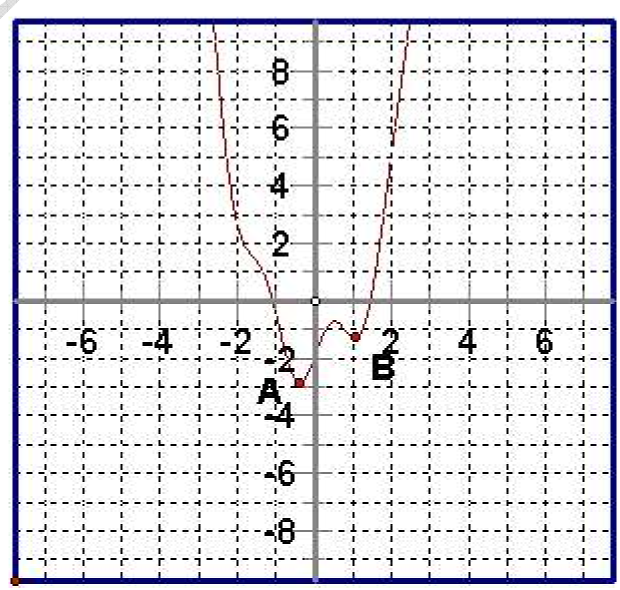
For each graph below, find an approximate value for the slope of the tangent at the point A by using a series of secants with A as one endpoint.

Compare and describe the instantaneous rate of change at point A and point B. Explain your reasoning.









Unit 1: Day 3: Exploring the Concept of Limit — Grade 12

75 min

Math Learning Goals
- Explore the concept of a limit by investigating numerical and graphical examples and explain the reasoning involved
- Explore the ratio of successive terms of sequences and series (use both divergent and convergent examples)
- Explore the nature of a function that approaches an asymptote (horizontal and vertical)

Materials
- BLMs 1.3.1-1.3.7
- large grid paper
- graphing technology
- 8.5 × 11 paper

Assessment Opportunities

Minds On…

Small Groups → Exploration/Discussion

In groups of three using one piece of 8.5 × 11 paper, guide students through the following exploration.

Each group of three will divide their paper in the following manner. Divide the paper into 4 equal pieces and each group member takes one piece.

What fraction of the paper does each group member have?

Divide the remaining piece into four equal pieces and each group member takes one piece.

What fraction of the paper does each group member have?

If this process continues indefinitely, how much of the paper will each person have? Explain your reasoning.

How does the graphical representation of the data help to visually explain the concept of the limit?

Whole Class → Instruction

Introduce the vocabulary of "limit" and "infinite sequence" and "infinite series."

GSP sketch Paper_Cut.gsp can be used to simultaneously plot the data points. Make use of an interactive whiteboard if available.

Students need access to graphing technology and/or large grid paper for each.

Action!

Small Groups → Investigation – Taking it to the Limit

Learning Skill/Observation/Mental note: Observe students to identify Teamwork and Work Habits.

Each group works on one of the following three investigations. Circulate during the task and provide direction as necessary.

- **Investigation 1:** Using BLM 1.3.1 & 1.3.4(teacher), students investigate the concept of a limit using series.
- **Investigation 2:** Using BLM 1.3.2, students investigate the concept of a limit using the sequence of ratios of successive terms the Fibonacci sequence.
- **Investigation 3:** Using BLM 1.3.3 & 1.3.5 (teacher), students investigate the concept of a limit using the behaviour of a function near an asymptote.

Mathematical Process of Lesson Focus: Reasoning, Representing, Students reason and to make connections between different representations of data and the concept of a limit.

Choose examples from BLM 1.3.4 & 1.3.5 for investigations 1 & 3. Further information about Fibonacci can be found at: http://www-history.mcs.st-andrews.ac.uk/Biographies/Fibonacci.html http://evolutionoftruth.com/div/fibocalc.htm

http://www.mathcentre.ac.uk/staff.php/mathematics/series/limits/resources/resources/366

Consolidate Debrief

Whole Class → Presentations and Discussion

Select groups to present their findings from one of their examples. Highlight process and findings.

Using BLM 1.3.6, present a series of graphs showing different representations (data; graphs of discrete data points; graphs of smooth, continuous functions) and summarize the concept of a limit for each.

Explore further scenarios with series such as 1 – 1 + 1 – 1 + … .

Differentiated Instruction

Home Activity or Further Classroom Consolidation

Using BLM 1.3.7, assign tasks on observed student success with the investigations.

1.3.1: Taking it to the Limit

In your group, investigate the two examples assigned using the outline below. You may wish to use graphing software to help you with your analysis. Be prepared to present the findings of your group with rationale.

Analysis:

To analyse the existence of a limit of these series, create a sequence off partial sums

$S_1, S_2, S_3, \ldots, S_{10}$, where S_1 represents the sum of the first term
S_2 represents the sum of the first two terms
S_3 represents the sum of the first three terms
.
.
.
S_{10} represents the sum of the first ten terms

Term Value	Partial Sum for Series One	Partial Sum for Series Two
1		
2		

Select a tool to create a data plot, where n (the sum number) is the independent variable and S_n is the dependent variable. Sketch the data plot on the grid provided for one of the sequences for which a limit exists.

Summary:

1) State your series ____

2) State the sequence of sums ____

3) The behaviour of our sequence is

4) We reached this conclusion because

1.3.2 Investigating Ratios in the Fibonacci Sequence

http://www-history.mcs.st-andrews.ac.uk/Biographies/Fibonacci.html

The Fibonacci sequence is an example of a *recursive* sequence. Each number of the sequence is the sum of the two numbers preceding it. Formally, this can be written:

$$F(1) = 1$$
$$F(2) = 1$$
$$F(n) = F(n-1) + F(n-2)$$

1. Complete the following table computing the ratios of consecutive terms **correct to 5 decimal places**.

Fibonacci Number $F(n)$	**Ratio of Consecutive Terms** $\dfrac{F(n)}{F(n-1)}$
1	
1	$t_1 = \frac{1}{1} =$
2	$t_2 = \frac{2}{1} =$
3	$t_3 =$
5	
8	
13	

2. Create a new sequence from the ratios obtained in Question 2, i.e. $t_1 = 1$, $t_2 = 2$, $t_3 = 1.5$ …

   1, 2, 1.5, ______, ______, ______, ______, ______, ______, ______, ______, ______, ______

3. Using graphing technology, create a plot of the sequence in Question 2, where the term number is the independent variable and the value of the ratio is the dependent variable. Sketch the graph on the grid provided.

4. Does the sequence of ratios of Fibonacci numbers have a limit? Justify your answer.

1.3.3: Taking it to the Limit

In your group, investigate the two examples assigned using the outline below. You may wish to use graphing software to help you with your analysis. Be prepared to present findings of your group with rationale.

Function Analysis

Using the tool of your choosing, create a graph for the given domain. Sketch the graph on the grid provided for each of the functions assigned.

Summary

1) State your function. 2) Describe the behaviour of the function over the given domain. 3) State the limit if one exists. Explain.	1) State your function. 2) Describe the behaviour of the function over the given domain. 3) State the limit if one exists. Explain.

1.3.4: Sample Exploration Questions for BLM 1.3.1 (Teacher)

Each group of students receives one series from group A and one series from group B.

Group A:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}+\ldots$$

Examine 10 sums

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\ldots+\frac{1}{n^3}+\ldots$$

Examine 10 sums

$$\frac{1}{2}+1+1.125+1+\ldots+\frac{n^2}{2^n}+\ldots$$

Examine 10 sums

Group B:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

Examine 10 sums

$$3-9+9-12+\ldots+(-1)^{n+1}(3n)+\ldots$$

Examine 10 sums

$$2+4+6+8+16+\ldots+2^n+\ldots$$

Examine 10 sums

1.3.5: Sample Exploration Questions for BLM 1.3.3 (Teacher)

Each group of students receive one series from group A and one series from group B.

Group A:

$f(x)=\frac{1}{x}+4$ — Examine end behaviour as x becomes large.

$f(x)=-\frac{1}{(x-2)}$ — Examine behaviour as $x \to 2$ (i.e., x gets close to 2) beginning with values $x=1.5$ and incrementing by 0.1

$f(x)=\frac{-3(x-3)}{x}+5$ — Examine end behaviour as x becomes large.

Group B:

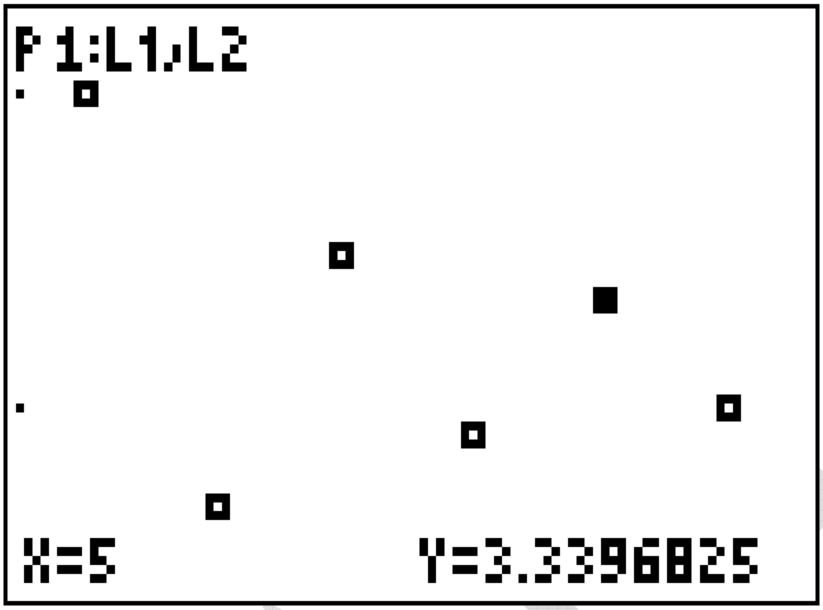
$f(x)=\sqrt{x-3}$ — Examine end behaviour as x becomes large.

$f(x)=-2x^2+x-4$ — Examine end behaviour as x becomes large.

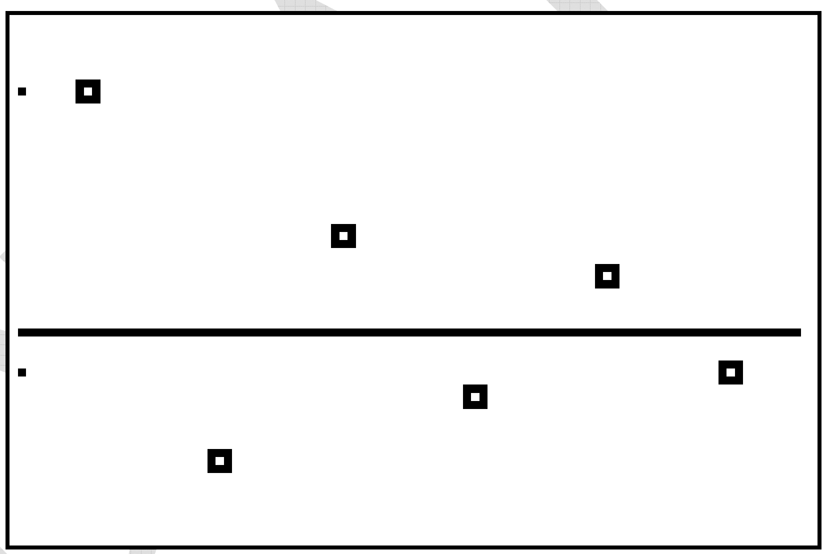
$f(x)=\frac{-3(x-3)}{x}+5$ — Examine behaviour as $x \to 0$ (i.e., x gets close to 0) for values of x beginning with $x=-1$ and incrementing by 0.1

1.3.6: Sample Slides for Debrief (Teacher)

1)

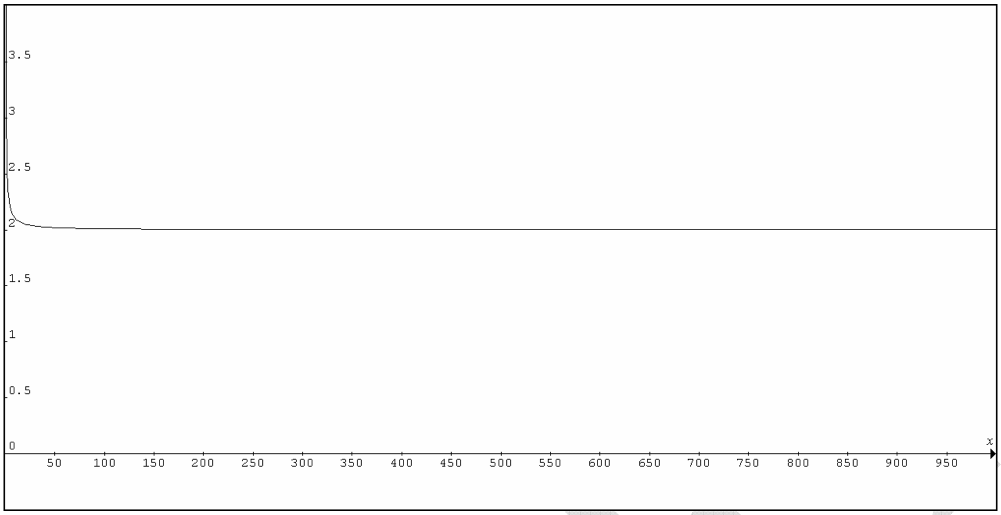


2)



1.3.6: Sample Slides for Debrief (Teacher) (continued)

3) $\lim_{x \to 0} f(x)$ for numbers greater than 0 as compared to $\lim_{x \to \infty} f(x)$



X	Y1
0	ERROR
50	2.02
100	2.01
150	2.0067
200	2.005
250	2.004
300	2.0033

X=0

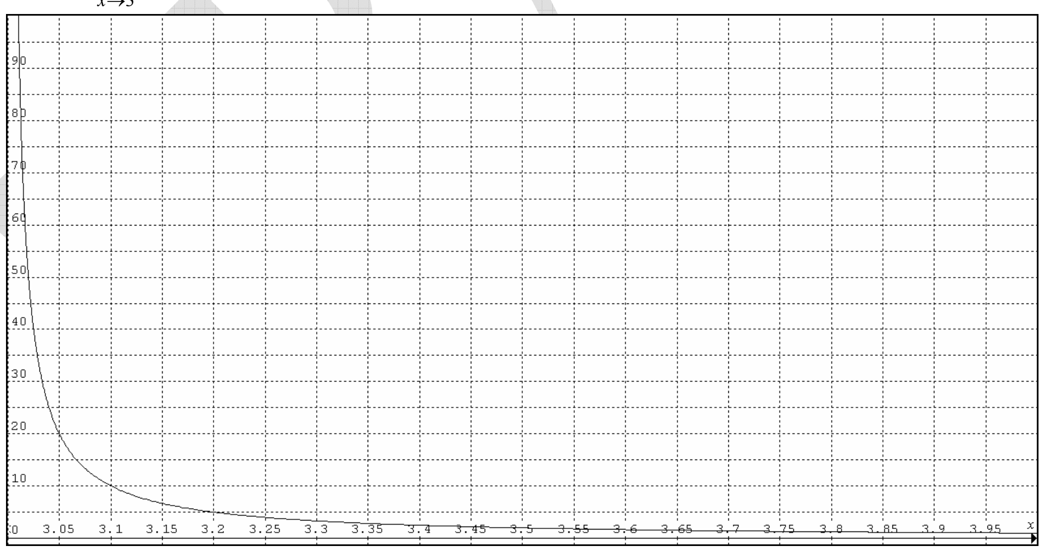
X	Y1
700	2.0014
750	2.0013
800	2.0013
850	2.0012
900	2.0011
950	2.0011
1000	2.001

X=1000

X	Y1
350	2.0029
400	2.0025
450	2.0022
500	2.002
550	2.0018
600	2.0017
650	2.0015

X=650

4) Looking at $\lim_{x \to 3} f(x)$ for numbers greater than 3



X	Y1
4	1
3.95	1.0526
3.9	1.1111
3.85	1.1765
3.8	1.25
3.75	1.3333
3.7	1.4286

X=4

X	Y1
3.65	1.5385
3.6	1.6667
3.55	1.8182
3.5	2
3.45	2.2222
3.4	2.5
3.35	2.8571

X=3.65

X	Y1
3.3	3.3333
3.25	4
3.2	5
3.15	6.6667
3.1	10
3.05	20
3	ERROR

X=3

1.3.7: Home Activity Ideas (Teacher)

Idea 1:
Consider the repeating decimal 0.9999999…. Represent this decimal as a fraction. Explain the result in terms of limits.

Idea 2:
Does a limit exist? A staircase is constructed that has a vertical height of 4 units and a horizontal length of 4 units. Each step has a length of 1 unit horizontally and 1 unit vertically, so there are four stairs. The total of the vertical and horizontal distances is 8 (4 x 1 up and 4 x 1 across). Now, put in twice as many stairs by making each step half as long and half as high. What is the total of the vertical and horizontal distances? Continue to double the number of steps by halving the length and height of each step. Describe the limiting process taking place as the doubling continues, forever.

Idea 3:
Describe the limit of the following process. (Do not look for a numerical solution – rather look for a descriptive solution): Inside a unit circle is inscribed an equilateral triangle. Inside the triangle is inscribed a circle. Inside this circle is inscribed a square, into which is inscribed a circle. Inside this circle is inscribed a regular pentagon, into which is inscribed a circle…

Idea 4:

What is the value of the fraction $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1\ldots}}}}$ as the division continues? Compare the result with that of Investigation 2 (BLM 1.3.2).

Idea 5:
Investigate the area and perimeter of the Sierpinski Triangle as the number of iterations increases.

Idea 6:
Investigate the area and perimeter of the Koch Snowflake as the number of iterations increases.

Unit 1: Day 4: Calculating Instantaneous Rates of Change Numerically — Grade 12

75 min

Math Learning Goals

- Connecting the average rate of change of a function to the slope of the secant using the expression $\dfrac{f(a+h)-f(a)}{h}$
- Connecting the instantaneous rate of change of a function to the slope of the tangent using the expression $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

Materials

- BLM 1.4.1
- BLM 1.4.2
- Computer and data projector
- Geometer's Sketchpad®

Assessment Opportunities

Minds On…

Whole Class → Discussion

Activate prior knowledge of function notation, secant lines, slopes of line segments and average rate of change from MHF4U and previous lessons in this unit.

Use GSP sketch **Slope Secant.gsp** to develop the general expression for the slope of a secant line $\dfrac{f(a+h)-f(a)}{h}$.

BLM 1.4.1 can be used if no access to Geometer's Sketchpad® is possible.

The first page of the GSP® sketch determines the slope of the secant line numerically and the second page develops the general expression.

Action!

Pairs → Investigation

Students work in pairs on BLM 1.4.2 to develop the understanding that the slope of the secant becomes the slope of the tangent as h approaches zero.

Mathematical Process Focus: Connecting and Communicating

Curriculum Expectation/Observation/Mental Note: Observe students understanding of the connection between the slope of the secant and the slope of the tangent.

Consolidate Debrief

Pairs → Think/Pair/Share

Using the consolidating questions on BLM 1.4.2, student pairs use the think/pair share literacy strategy to consolidate understanding of concepts.

Summary:
When h approaches zero the secant line becomes a tangent line. To find the slope of the tangent line for $f(x)$ at $x = a$ you must evaluate $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

Exploration
Application

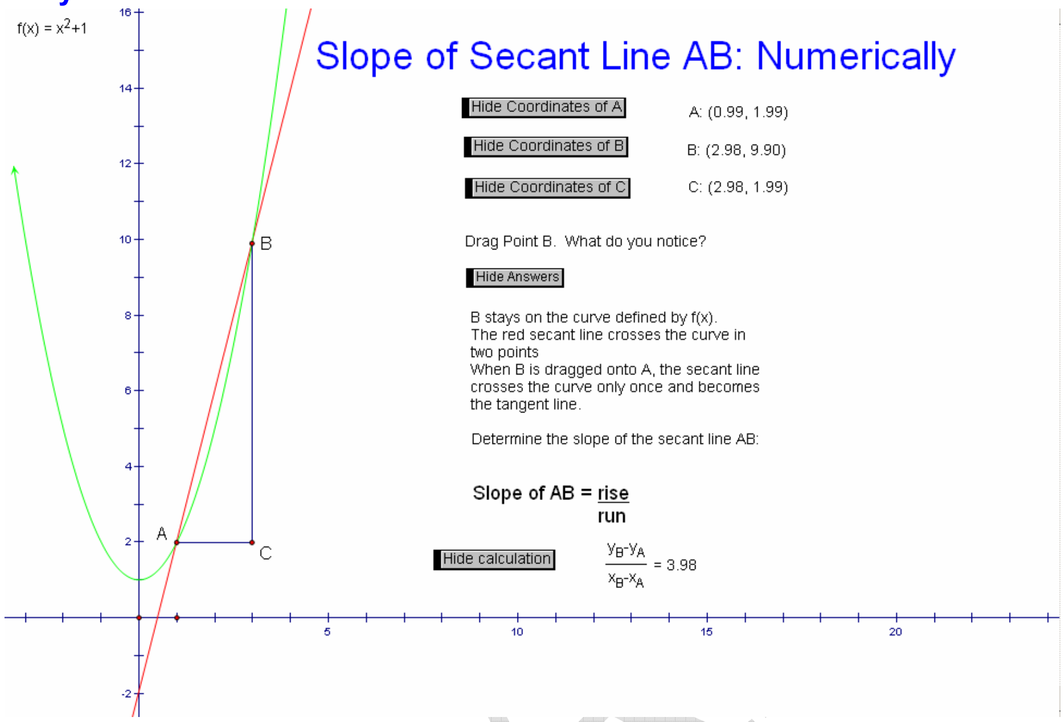
Home Activity or Further Classroom Consolidation

Complete practice questions from BLM 1.4.2

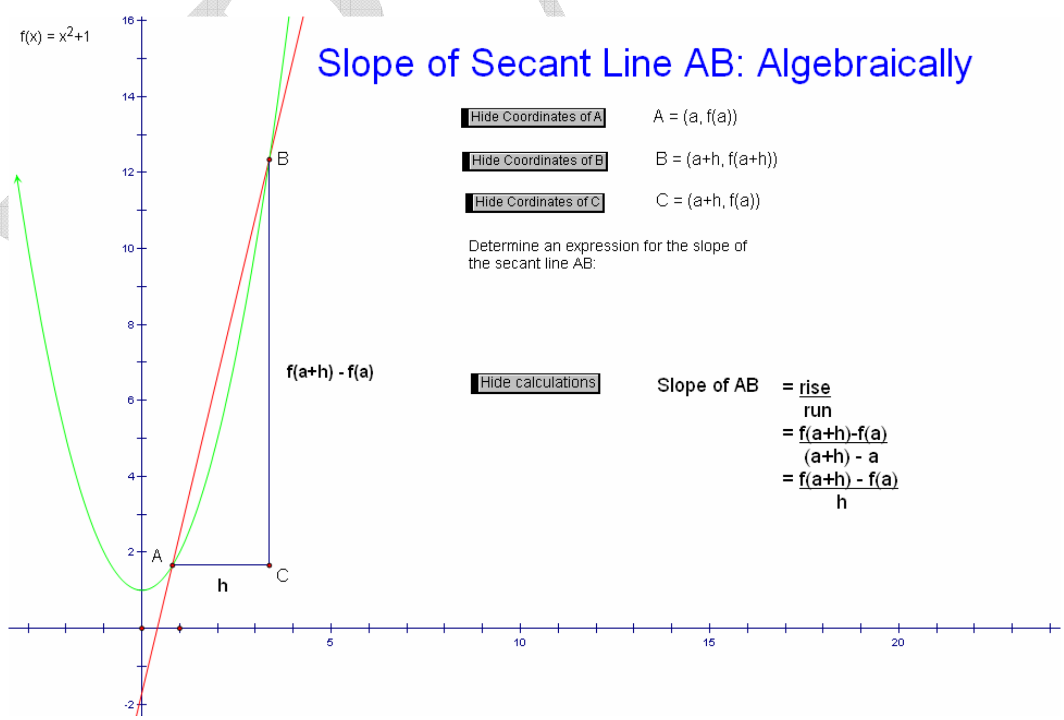
1.4.1 Secant Slope (Teacher)

Geometer's Sketchpad® Sketch: Secant Slope.gsp

Numerically:



Algebraically:



1.4.2: Determining Numerically the Instantaneous Rate of Change

With a partner you will determine the slope of secant lines from a point on a curve to another point where the x value is **h** units away from the original x value. You will calculate slopes of secants for smaller and smaller values of **h**.

Method 1: Substitute into the Slope of a Secant Expression

Function: $f(x) = x^2$ Value of a: 3

	h	$a + h$	$(a, f(a))$	$(a + h, f(a + h))$	$\dfrac{f(a+h)-f(a)}{h}$
3	1	4	(3, 9)	(4, 16)	7
3	0.1				
3	0.01				
3	0.001				
3	0.0001				

Method 2: Substitute into the Slope of a Secant Expression

Function: $f(x) = x^2$ Value of a: 3

$$\frac{f(a+h)-f(a)}{h}$$
$$=\frac{f(3+h)-f(3)}{h}$$
$$=\frac{(3+h)^2-(3)^2}{h}$$
$$=\frac{6h+h^2}{h}$$
$= 6 + h$, h **not equal to zero**

	$\dfrac{f(a+h)-f(a)}{h}=6+h$
1	7
0.1	
0.01	
0.001	
0.0001	

1.4.2: Determining Numerically the Instantaneous Rates of Change

(continued)

Consolidating Questions:

What value is the slope of the secant line approaching as h gets smaller and smaller? What does this value represent?

What do you notice about the results of the two methods?

Explain why the slope of the secant line is changing as the value of h decreases to zero.

Which method allows you to find the slope of the tangent to any point for any function?

Extra Practice:

Repeat the procedure of Method 1 and Method 2 to determine the slope of the tangent line to the following functions at the given value of $x = a$.

a) Function: $f(x) = x^2$ Value of a: 5

b) Function: $f(x) = 2x^2$ Value of a: 4

c) Function: $f(x) = x^3 + 4$ Value of a: 2