▲ Supporting Algebraic Thinking: Prioritizing Visual Representations

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For a number of years, mathematics education researchers have been promoting the introduction of algebraic concepts in elementary and middle schools (Warren & Cooper, 2006; Greenes et al., 2001; Kieran & Chalough, 1993; Kieran, 1992, 1991). There is a belief that an early introduction to algebra may ease the transition from arithmetically based instruction in elementary school to formal algebraic instruction in high school (Kieran, 1992). As part of this new initiative, mathematics education researchers have been investigating how young children can think algebraically when working with patterns (Yerushalmy & Shternberg, 2001; Noss & Hoyles, 2006; Mason, 1996; Lee, 1996).

In this article, I outline the results of a three-year research study that tracked the development of algebraic thinking in students from Grade 4 to Grade 6. Although I was working with elementary students, the concepts they considered were well beyond the current Ontario elementary curriculum expectations and incorporated concepts formally taught in the intermediate and senior grades. In the following sections, I introduce a sequence of activities that support students—from working with linear growing patterns, patterns whose terms grow by a constant value, to solving linear equations.

One of the goals of my work has been to introduce students to various representations of linear relationships. Understanding the connections among representations underpins the ability to predict changes in one representation based on transformations of the other.

The final sequence of lessons can be roughly divided into three parts. Part 1 includes unordered tables of values integrated with activities involving linear growing patterns. Part 2 introduces another visual representation, graphical representations. In part 3, students compare lines on a graph as precursory understanding for solving equations of the form $ax + b = cx + d$. These three parts were designed to help students transition from the expectations outlined in the Ontario junior and intermediate curriculum to the expectations outlined in the senior curriculum.

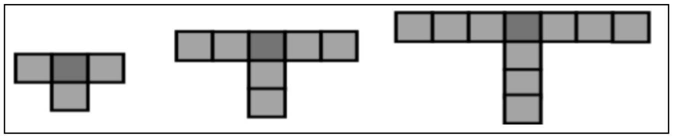
Grades 4-8	Grade 9
• recognize, describe, generalize patterns • use graphs, tables, and verbal descriptions to recognize relationship • understand variables as changing quantities • represent patterns algebraically and make predictions	• identify properties of linear relationships • investigate relationships between two variables • understand the characteristics of a linear relationship • connect representations—tables of values, graphs, and equations • understand the formula $y = mx + b$

In the sections below, I will review some of the well-documented difficulties students encounter as they develop their algebraic reasoning. I will then outline the components of the curriculum developed in response to these difficulties.

Patterns and Algebraic Thinking

In elementary curricula, algebraic understanding is introduced through working with patterns. Patterns offer a tremendous opportunity for students to explore some fundamental algebraic concepts. For example, patterns support students' abilities to generalize. Patterns also offer students a way of concretely exploring the idea of systematic variation between two sets of data.

The expectation is that working with patterns will support students to discover and articulate mathematical structure. Students working with patterns are expected to be able to describe the pattern, extend the pattern, and identify the underlying structural rule (generalize).



For instance, in the pattern above, most students are able to recognize and describe the pattern as "you add three more light square tiles each time." Students are also able to extend the pattern— the next "T" would have one dark square tile and four light square tiles on each of the three sides. The difficulty arises when students are asked to predict how many square tiles would be needed far down the sequence. For the tenth term, students might rely on drawing the 4th, then the 5th, then the 6th, and so on, adding one square tile to each arm as they go. Asking for the 100th term highlights the problem of this method. Students also cannot devise a general rule that would allow them to predict the number of square tiles for any term of the pattern. The problem with the rule "add three each time" is that for any term number, you need to know the number of square tiles in the preceding term. This is known as recursive reasoning. With recursive reasoning, only the variation in one data set—in this case, the number of tiles—is considered.

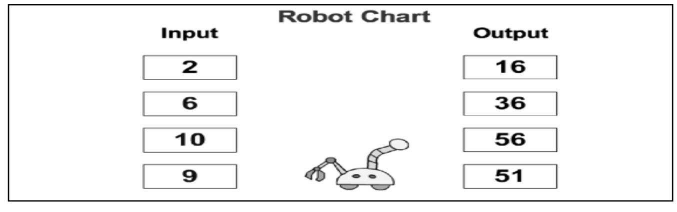
The pattern above can also be represented using an ordered table of values. This is the strategy that many students use to enable them to consider both the term number and the number of tiles at each iteration of the pattern.

Term Number	Number of Tiles
1	4
2	7
3	10

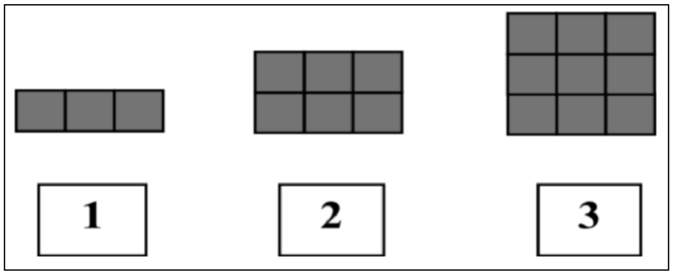
Ultimately, students learn strategies to figure out the number of tiles from the term number, but the focus of the study was to develop those strategies with conceptual understanding.

Curriculum Part 1

The activities in this part of the curriculum, (i.e., the lesson sequence developed) introduce students to linear relationships via unordered tables of values and linear growing patterns. Unordered tables of values are introduced using a "function machine" type activity (Willoughby, 1997) called the Robot Game. Teachers and students take on the role of a robot that transforms input numbers to output numbers by following a linear rule of the form $y = mx + b$. Students call out input numbers at random, which ensures that the resulting "robot chart" is unordered. Students are thus forced to look across the two data sets (input and output numbers) in order to determine the pattern rule. The rule is expressed as, for example, "the output number is equal to the input number times five plus six" and written as output=input x 5 + 6—which is structurally similar to $y = mx + b$.

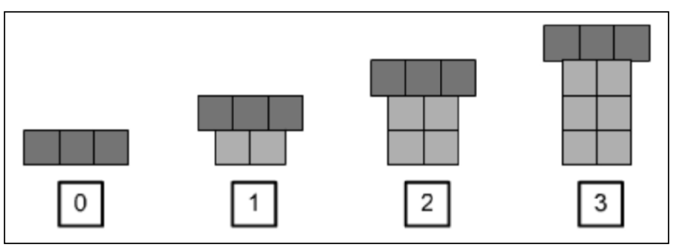


This numeric pattern activity is interspersed with activities involving linear growing patterns. Students are first introduced to the pattern below, built with square tiles and including “position cards” at each iteration.



In this activity, students can describe and extend the pattern as “add three more.” However, when asked for the number of tiles for the 10th position, most students correctly answer that there would be 30 tiles because 10 times 3 is 30. Students make the connection between the ordinal position number of each term of the pattern, and the number of tiles in that term. The position number is not a label, it is a quantity—the position numbers represent one set of data and the tiles represent the other set of data, and the pattern rule articulates how the two sets co-vary. In this case, unlike the case where recursive reasoning is used to determine the number of tiles for a far term or position, or for *any* position, students can use the pattern rule that describes the relationship between the position number and the number of tiles—“the number of tiles is equal to the position number times three.”

Students then work with more complex patterns like the one below. Here, the pattern rule can be articulated as “the number of tiles is equal to the position number times two plus three” and written as “number of tiles = position number x 2 + 3.”



The multiplier, x 2, is represented by the light tiles that grow by 2. The constant, +3, is represented by the three dark tiles that “stay the same” at each position. Notice that at the “zeroth” position, the multiplier is represented by 0 tiles, and the constant by the three dark tiles.

By engaging in a number of Robot Game and pattern-building activities, students learn that linear rules can be expressed as the relationship between input and output numbers in unordered tables of values, and as the relationship between the number of tiles for each position of a linear growing pattern. They are able to consider particular instances of a linear growing pattern and formulate a general rule. Students also make connections between the Robot Game activities and the pattern-building activities, and the different ways to express the rules, so that “output = input x 3 + 2” is linked to “number of tiles = position number x 3 + 2.” The students recognize that both are representations of the underlying mathematical rule. The following is an excerpt from a transcript of a classroom lesson, during which a Grade 6 student explained the connections he saw between the Robot Game and building linear growing patterns.

> *In pattern building, the input part is the position number and the output part is like the pattern you build. So in the middle is the operation or the rule you have to use. So it’s kind of like the Input/Output (Robot Game). In the Input/Output you have to use the Input, do the rule and then you get the Output. The same with this one (pointing to pattern), you have to use your position number, do the rule, and get your answer and make the pattern.*

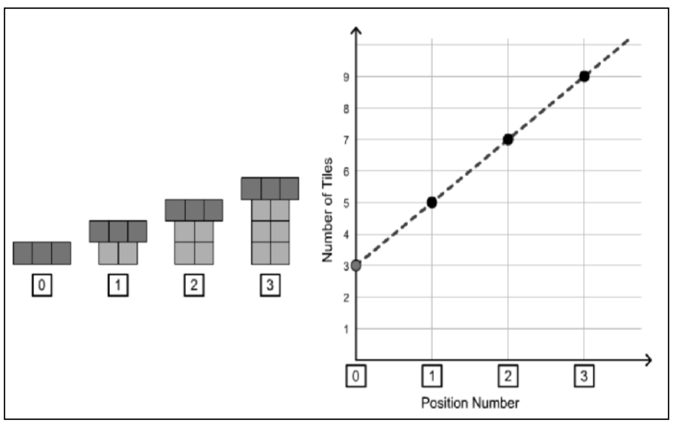
Graphs and Algebraic Thinking

Numerous researchers have documented the difficulties students encounter when considering graphical representations of linear relationships (e.g., Evan, 1998; Moschkovich, 1996, 1998, 1999; Brassel & Rowe, 1993; Yerushalmy, 1991). Many students find it difficult to connect the algebraic equation $y = mx + b$ with the graph and do not realize that the multiplier of the independent variable is responsible for the angle (or slope) of the line, and that the constant is responsible for the y-intercept (Bardini & Stacey, 2006). Students find it difficult to predict how changing the multiplier or the constant in the equation will affect the graph. For instance, most students predict that the line for the equation $y = 5x + 5$ will be both higher and steeper than the line for the equation $y = 5x$; students are unaware that these two parameters (m and b) are independent (Moschkovich, 1996). Because the two domains of solving algebraic equations and constructing and interpreting linear graphs are often taught separately, students often treat the algebraic and graphical representational domains as if they were independent.

Another difficulty some students have is that they tend to focus on individual points (like the y-intercept), but do not recognize the line as a representation of linear growth (Leinhardt, Zaslavsky & Stein, 1990).

Curriculum Part 2

When engaged in activities from the second part of the curriculum, students learn how to construct linear graphs based on their understanding of pattern rules. At this point, the students use only the upper-right quadrant of the coordinate plane, since the pattern rules include only positive integers for both the multiplier and constant. The position number cards of the growing patterns are mapped onto values along the x-axis. The total number of tiles in each position is represented by values along the y-axis. One point on the y-axis represents the number of tiles that would be at the "zeroth" position of a growing pattern, the value of the constant, which is graphed as the y-intercept. When constructing the graph, students are given rules, such as "number of tiles = position number x 2 + 3," and asked to build geometric patterns based on the rule. They are then asked to calculate the total number of tiles at each position of the pattern and draw a dot on the graph to represent how many tiles are at each of the position numbers built. Students therefore graph "ordered pairs"—position number, number of tiles—without being explicitly asked to do so.



Students then build and create a graph of the 0th, 1st, 2nd, and 3rd position for three pattern rules that have different multipliers and the same constant ($y = x + 1$, $y = 3x + 1$, $y = 5x + 1$), and for rules that have similar multipliers and different constants ($y = 3x + 1$, $y = 3x + 3$, $y = 3x + 5$). For each set of rules, students are asked to predict what the graph would look like (e.g., parallel lines, lines with different steepness). The students build all three patterns and graph them, and note similarities and differences between the pattern rules, the patterns, and the graphs, both within and between the two sets of rules. Thus, students are given the opportunity to explicitly consider how a change in one representation, the numeric/symbolic rule, affects both the linear growing pattern and the graph.

In the following transcript, a student discusses the graph of pattern rules that have the same multiplier, but different values for the constant. The student is able to reason about the connection between the two parameters of the pattern rules and the parallel lines on the graph.

> *The lines are parallel to each other and that was my prediction—that they would have the same steepness, but have different heights. The multiplier decides the steepness. If the multiplier is big, like times 9, then the line goes steep, but if it's lower, like times 3, it's not going to be that steep. The constant decides where the line starts—so, like the height of the line on the graph. If you have the same multiplier, all the lines have the same steepness, and the constant decides the height of the line.*

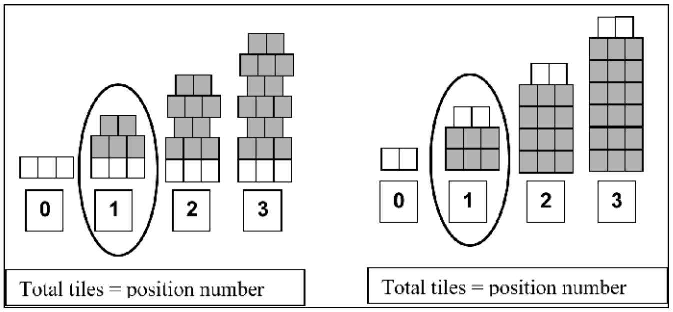
Students realize that graphs model the rate of change of the growing patterns, and that the higher the multiplier, the more tiles are added to each successive position in their pattern, and the steeper the slope of the line. Students also developed an understanding that the constant is represented by "where the line starts" on the y-axis (y-intercept), since the number of tiles at the "zeroth" position of a pattern is graphed at the vertical axis, and only the value of the constant is represented (Beatty, 2007).

Curriculum Part 3

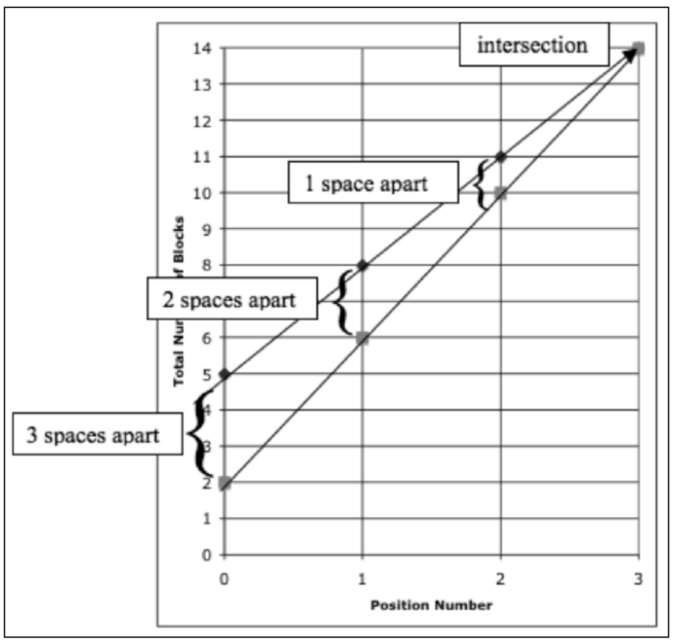
In the third part of the curriculum, to provide a conceptual background for solving equations of the form $ax + b = cx + d$, the lesson sequence is designed to support students to think about intersecting lines as representing the relationship between two linear rules.

Students are given pattern rules and asked to predict the position at which both rules would have the same number of tiles. They are then asked at what x-value on the graph the lines would intersect.

When comparing two rules, such as $5x + 3$ and $6x + 2$, students initially identify that both rules would have 8 tiles at position 1 (if they were linear growing patterns) and that this would be represented by lines that intersect at (1,8) on the graph.



Students then discover that when considering any two pattern rules that have a difference of 1 for the multipliers, the difference between the constants is the x-value at which the lines will intersect. For instance, $4x + 2$ and $3x + 5$ have y-intercepts that are 3 spaces apart on the graph, and the lines come together by one space each time.



Knowing how far apart 2 rules "start"—i.e., based the values of the constants represented by the y-intercepts—and knowing the rate at which the lines "come together" on the graph—i.e., knowing that if the multipliers differ by 1, the lines come together by one space at each successive x-value—allows for the prediction of the x-value at which the lines will intersect.

The students further generalize this understanding, realizing that you can work out the point of intersection if you know where any lines start—the difference between the constants—and compare the rate at which they come together—by comparing the value of the multipliers. For example, $3x + 6$ and $5x + 2$ start out 4 spaces apart on the graph (the y-intercepts are 4 spaces apart), but come together by two spaces at each successive x-value. Therefore, you can figure out that the lines will meet, or intersect, by x-value 2.

This understanding can be used to solve equations of the form $ax + b = cx + d$. Students are given rules in standard algebraic notation and asked, "At what position would the lines for these rules intersect?" At this point, the graph is no longer the site for problem solving, but instead, is used as a tool for checking and justifying solutions. For example, John was given the equation $2x + 16 = 5x + 1$ and asked to predict the point of intersection, which is, in effect, solving for x.

John: Well, I would see that the difference between these two (the multipliers $2x$ and $5x$) is 3 and that the difference between these two (constants +16 and +1) is 15. I know that 15 divided by 3 is 5, so I think it's going to intersect on the 5th position.

Teacher: How would you check?

John: Try it out. So 5 x 2 is 10 plus 16 is 26, and 5 times 5 is 25 plus 1 is 26.

Teacher: What does it mean when you get 26 for both rules?

John: Um, that's the amount of each pattern and that's where they would intersect.

Teacher: But I thought you said they intersect at 5?

John: The 5th position! [pointing to the x-value 5 on the graph] At the 5th position, they would both equal 26 tiles—that's like the number they would intersect on. So it would be [drawing a dot on the graph at (5, 26)] they would both end up there.

In his explanation, John integrates multiple representations by making a reference to pattern building, that at the "5th position, they would both equal 26 tiles," and the graph by stating that "they would both end up" at (5, 26).

If we think of $ax + b$ and $cx + d$ as two pattern rules for which the value of the multiplier and the constant are different, then the solution to the equation $ax + b = cx + d$ is analogous to finding the x-value at which the lines of the two rules will intersect (or the position number at which both patterns will have the same number of tiles). To determine how far apart lines "start" on the y-axis, students find the numeric difference between the values of the constant, or $(d - b)$. To find the rate at which they "come together," students find the difference between one multiplier and the other $(a - c)$. To find the position number (x), they divide "how far apart they started" by "the rate at which they come together" or $x = (d - b) \div (a - c)$. Students demonstrate a conceptual understanding of why they carry out the operations of subtraction and division, rather than just memorizing an algorithm.

Conclusion

Current research indicates that there is a need for instructional models that help students understand the connections among representations of linear relationships (e.g., Greenes et al., 2007). The instructional sequence described in this paper has been shown to support Grades 4, 5, and 6 students in developing their understanding of generalizing, mathematical structure, and the connections among representations of linear relationships. Results indicate that these young students are capable of engaging in algebraic thinking beyond current provincial elementary curriculum expectations and can consider concepts from the intermediate and senior mathematics curricula.

In particular, the prioritization of visual representations seems to support students' emergent understanding of linear relationships. Understanding a concept presupposes the ability to recognize that concept in a variety of representations and the ability to handle the concept flexibly within different representational systems. By utilizing students' understanding of patterns, the connections between equations and graphic representations were made as transparent as possible. Students were able to see how changing a pattern rule resulted in changes in the linear growing pattern, which then led to the ability to predict how transforming a rule would also transform the graph of that rule. This then led to an initial understanding of comparing lines on a graph and using that understanding to solve equations $ax + b = cx + d$.

Preliminary results indicate that this may also be a useful approach for students in Grades 7 and 8 (Bruce, Ross, Beatty & Sibbald, 2010). However, more research is required to determine whether an approach to teaching linear relationships that integrates multiple representations, and prioritizes visual representations, can support the transition of middle school students to formal algebraic instruction.

On the OAME website, CLIPS materials based on the ideas in this article are available for use with students.

References

Bardini, C., & Stacey, K. (2006). Students' conceptions of m and c: How to tune a linear function. In Novotna, J., Moraova, H., Kratka, M. & Stehlikova, N. (Eds.), *Proceedings of the 30th conference of the international group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Prague, Czech Republic: Psychology of Mathematics Education.

Beatty, R. (2007). Young students' emergent understanding of linear functions: Linking symbolic, graphic and geometric representations. Paper presented at the annual meeting of the American Educational Research Association, Chicago.

Beatty, R., & Moss, J. (2006). Multiple vs. numeric approaches to developing functional understanding through patterns – affordances and limitations for grade 4 students. In Altorre, S., Cortina, J.L., Saiz, M., and Mendez, A. (Eds.), *Proceedings of the twenty-eighth annual meeting of the Psychology of Mathematics Education, North American Chapter.* Merida, Mexico.

Brassel, H.M., & Rowe, M.B. (1993). Graphing skills among high school physics students. *School Science and Mathematics, 93*, 63–71.

Bruce, C., Ross, J., Beatty, & R., Sibbald, T. (2010). Trigonometry and linear growing patterns field tests 2009–2010: A mixed-methods study. Report prepared for the Ontario Ministry of Education.

Evan, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105–121.

Greenes, C., Cavanagh, M., Dacey, L., Findell, C., & Small, M. (2001). *Navigating through Algebra in Prekindergarten – Grade 2.* Reston, VA: National Council of Teachers of Mathematics.

Greenes, C., Chang, K.Y., & Ben-Chaim, D. (2007). International survey of high school students' understanding of key concepts of linearity. In Woo, J.H., Lew, H.C., Park, K.S., & Seo, D.Y. (Eds.), *Proceedings of the 31st conference of the international group for the Psychology of Mathematics Education* (Vol. 2, pp. 273–280). Seoul, Korea: Psychology of Mathematics Education.

Kieran, C. (1992). The learning and teaching of school algebra. In T.D. Grouws (Ed.), *The handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kieran, C. (1991). *Helping to make the transition to algebra. Arithmetic Teacher, 38,* 49–51.

Kieran, C., & Chalouh, L. (1993). Pre-algebra: The transition from arithmetic to algebra. In D. Owens (Ed.), *Research ideas for the classroom: Middle grade mathematics* (pp. 179–198). New York: Macmillan.

Lee, L. (1996). An initiation into algebraic culture through generalization activities, in N. Bednarz, C. Kieran, & Lee, L. (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Dordrecht, Netherlands: Kluwer Academic Publishers.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & Lee, L. (Eds.), *Approaches to* *algebra: Perspectives for research and teaching* (pp. 65–86). Dordrecht, Netherlands: Kluwer Academic Publishers.

Moschkovich, J. (1999). Students' use of the *x*-intercept as an instance of a transitional conception. *Educational Studies in Mathematics, 37,* 169–197.

Moschkovich, J., (1998). Resources for refining conceptions: Case studies in the domain of linear functions. *The Journal of the Learning Sciences, 7*(2), 209-237.

Moschkovich, J. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *The Journal of the Learning Sciences, 5*(3), 239–277.

Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *International Journal of Computer Supported Collaborative Learning, 1*(4), 441–465.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics, 33*(2), 203–233.

Warren, E., & Cooper, T. (2006). Repeating and growing pattern rules: Relationships between geometric and numerical representations. Paper presented at the American Educational Research Association Annual Conference, San Francisco, April 7–11.

Willoughby, S. (1997). Functions from kindergarten through sixth grade. *Teaching Children Mathematics, 3* (February): 314–318.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning, 7*(1), 42–57.

Yerushalmy, M., & Shternberg, B. (2001). Charting a visual course to the concept of function. In Cuoco, A., & Curcio, F. (Eds.), *The roles and representations in school mathematics, National Council of Teachers of Mathematics 2001 yearbook.* Reston, VA: National Council of Teachers of Mathematics. ▲