Unit 4
Rate of Change Problems

Calculus and Vectors

Lesson Outline:

Day	Lesson Title	Math Goals	Expectations
1, 2, 3	*Rate of Change Problems* *(Sample Lessons Included)*	• Make connections between the concept of motion and the concept of the derivative in a variety of ways. • Make connections between graphical and algebraic representations and real-world applications • Solve problems in wide variety of contexts and interpret the results	B2.1 B2.2, B2.3
4, 5, 6, 7, 8	*Optimization Problems* *(Sample Lessons Included – Days 4 {without and with CAS} & 8)* **** New – Jan 08***	• Solve a variety of optimization problems given an algebraic model • Solve a variety of optimization problems requiring the creation of an algebraic model	B2.4
9	*Solve problems from data*	• Solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate results. • Revisit some of the rate of change and rate of flow problems from Unit 1	B2.5
10	*Jazz Day*		
11, 12, 13	*Summative Assessment for Units 3 – 4* *(Sample Lesson Included)*		

Unit 4: Day 4: Introduction to Optimization

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Minds On: 10 Action: 55 Consolidate:10 Total=75 min	**Learning Goal**: • Solve a variety of optimization problems given an algebraic model • Solve a variety of optimization problems requiring the creation of an algebraic model	**Materials** • Graphing calculators • 8.5x11 cardboard sheet • 8.5x11 paper • scissors • tape or glue • ruler • BLM 4.4.1 • BLM 4.4.2 •

Assessment Opportunities

Minds On…	**Whole Class → Discussion** Discuss the definition of optimization. Optimization - The procedures used to make a system or design as effective or functional as possible. The mathematical procedure: finding the maximum/minimum of a function. Have the students brainstorm ways that they can find if something is optimized for a variety of objects/situations. (i.e,. students can determine points and manually find which points give the maximum/minimum cost/area/volume/etc., find a formula if quadratic and find the vertex, use geometer sketchpad, graphing calculator, etc.) **Whole Class → Activity Instructions** Introduce the building a box activity to students by reading through BLM 4.4.1. Divide students into groups of 4 - 6.	Students can solve this problem in a variety of ways: graphing calculator, trial and error or optimization (taking the derivative) can be used to solve the problem. Other methods such as using Geometer Sketchpad are possible as well. Also, different sizes of paper/cardboard can be used so that each group has a different problem to solve.
Action!	**Small Groups → Experiment** Students can use graphing calculators, trial and error, or optimization to complete BLM 4.4.1. Students look at different methods to solve the problem and will discover that one method is more effective than other methods. Students need to think about their process and defend its merits by using calculators or other materials to support their ideas. The relationship between a slope of zero and maximum/minimum points should also be discussed. **Process Expectations/Observation/Rubric:** Observe and listen to students as they engage in problem solving and reasoning and proving. **Mathematical Process Focus: Problem Solving, Reasoning and Proving**	Continuing to add to the understanding of finding the derivative and how finding the maximum/minimum of something relates to optimization and a slope of zero.
Consolidate Debrief	**Whole Class → Discussion** Have students share their responses from BLM 4.4.1 and the steps they took to solve the problem with the class. Possible guiding questions: o What observations can you draw from your results numerically and graphically? o How does taking the derivative of the function help solve the problem?	
Practice	**Home Activity or Further Classroom Consolidation** • Complete BLM 4.4.2.	

4.4.1: Introduction to Optimization

PROBLEM:
You have been hired by Boxes R' Us to design a box in the shape of a rectangular prism that will hold miniature cubes. This open top box will be made from an 8.5 inch by 11 inch cardboard sheet. Your job is to determine the size of square that needs to be cut from each of the corners to maximize the volume of the box.

The box needs to be as large as possible so that it can hold the maximum amount of cubes but keep the cost of constructing the boxes at a minimum.

When you are done you must identify the dimensions of the squares that are to be cut from each corner and explain how you know that your box maximizes the volume.

STEPS:
Each group will be given only one sheet of cardboard.

1. Review the problem with your group.

2. Represent the problem by making a picture, chart, graph, list of knowns or givens, or any other form of framing the problem.

3. Form a hypothesis or prediction and/or plan a strategy for solving the problem.

4. Once you have shown the teacher that you understand the problem the teacher will give you the materials.

5. Look through your materials. Review your hypothesis and representation. Do they coincide with your materials? If not, change the hypothesis and representation and ask the teacher more questions about your activity.

6. Implement the strategy.

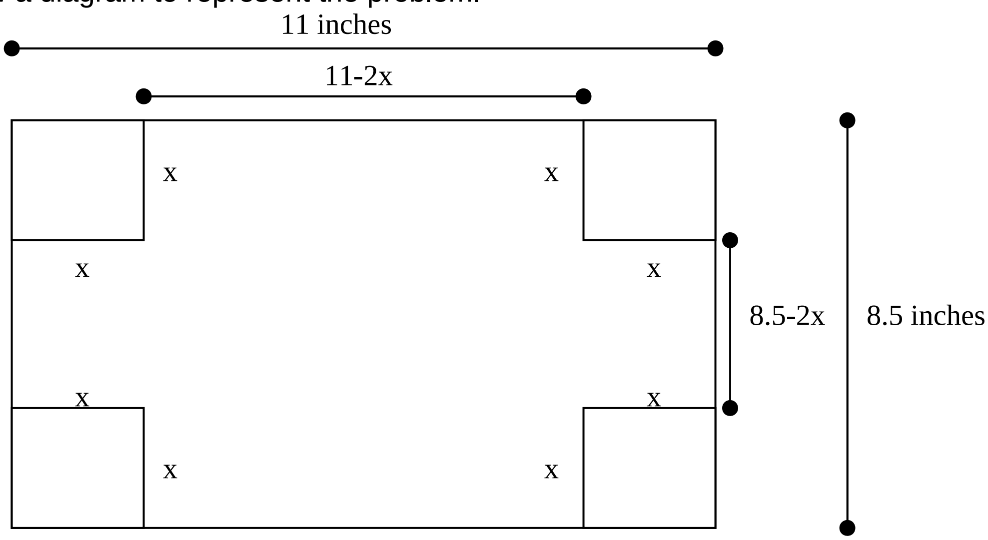
7. Evaluate the results and then confirm with the teacher by asking questions again. If the results do not make sense then return to step 1 and try again.

4.4.1: Introduction to Optimization (Teacher Notes)

This problem can be done in many ways. One method is by using Calculus. Another method is to use the graphing calculator to find the turning points. Trial and error can also be used to find the solutions.

<u>Calculus Solution:</u>

Draw a diagram to represent the problem.



According to the drawing x will represent the height of the box, $11-2x$ will represent the length of the box and $8.5-2x$ will represent the width of the box. Therefore volume of the box will can be represented by the equation $V = (11-2x)(8.5-2x)(x)$, where x has to be positive. Therefore $8.5-2x>0$, so $x<4.25$.

The expanded equation is $V = 4x^3 - 39x^2 + 93.5x$. A careful examination of the function expression (or a quick graph) shows that a maximum point will occur first (moving from left to right) and then a minimum point. The first derivative can be used here.

The derivative of $V(x) = 4x^3 - 39x^2 + 93.5x$ is $V^{/}(x) = 12x^2 - 78x + 93.5$.
Now set $V^{/} = 0$ since its slope is zero and solve for x using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

x = 1.585 and x = 4.915
The value of 1.585 for x will produce the maximum volume of the box (remember that x cannot be greater than 4.25).

Check: Use the second derivative function (If $V^{//}$(1.585) < 0, this indicates the volume function has a local maxima at this point)

$V^{/} = 12x^2 - 78x + 93.5$
$V^{//} = 24x - 78$
$V^{//}(1.585) = 24(1.585) - 78$
$V^{//}(1.585) = -39.84$

Therefore when x = 1.585 inches it produces the maximum volume of the box.

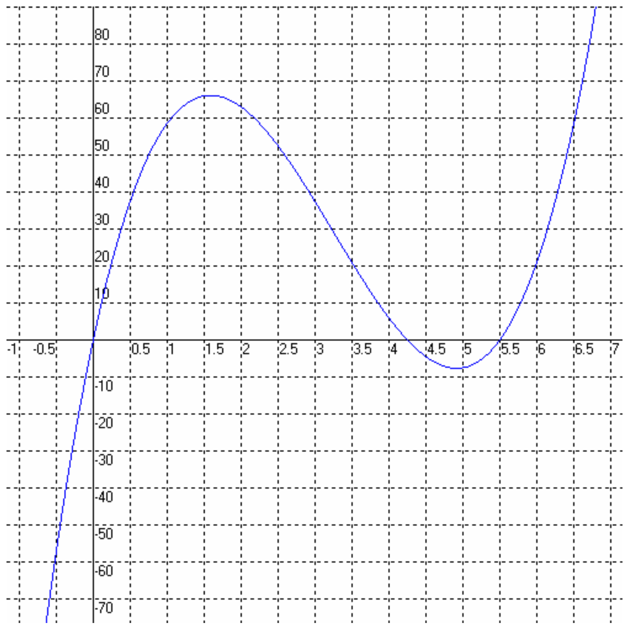
And the Maximum Volume $(V) = 4(1.585)^3 - 39(1.585)^2 + 93.59(1.585)$
$= 65.355\ in^3$

4.4.1: Introduction to Optimization (Teacher Notes continued)

<u>Using the Graphing Calculator:</u>

Proceed as in the first solution to get the equation: $V = 4x^3 - 39x^2 + 93.5x$ and its restrictions.

Enter the equation into the graphing calculator and graph it. Use the succession of buttons: 2nd, TRACE, Maximum, move point so that it is to the left of the highest point, hit Enter, then move the cursor to the right of the maximum point, hit Enter, Enter. The maximum point is at (1.585, 66.148).



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The x-intercepts may be used to find the maximum as well.

Set V to 0 and solve for x.

Need to factor the part in the brackets. Use quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Thus, $x = 0$, $x = 4.25$, $x = 5.5$

<u>Trial and Error Solution:</u>

A guess and check method is possible, though not the focus at this point in the course. The benefit of showing students a numerical approach at this point is its transferability to situations where the functions are "messy" or not known.

Set up a simple expression where x is the length of the cut-off square:

$V = (11-2x)(8.5-2x)(x)$ to find the dimensions of the inside of the box. Establishing the domain of x for this context at the outset is a useful strategy (as mentioned in the Calculus solution) to save time and effort. Using different values of x, starting at a value of 1 and working up in values, the student will find the greatest value of the expression, which is 66.147816 inches cubed when x is equal to approximately 1.59 inches.

4.4.2 : Introduction to Optimization Practice:

1. Determine the *x*-intercepts of the equation: $y = x^3 + x^2 - 12x$

2. Factor the expression: $x^2 + 4x - 21$

3. Determine the roots of the equation: $x^3 - 10x^2 + 17x + 28 = 0$

4. How are the roots, factors, and intercepts of an equation related?

5. Determine the maximum/minimum points of each function above.

Unit 4: Day 4B: The Box Problem with CAS (alternative lesson)

Minds On: 10 Action: 40 Consolidate:25 Total=75 min	**Math Learning Goals:** • Solve optimization problems algebraically and numerically.	**Materials** • BLM 4.4B.1 • BLM 4.4B.2 • Device with CAS (e.g., Nspire, TI-89, TI-92, Voyage2000, Maple)

Assessment Opportunities

Minds On…

Small Group → Placemat

Students brainstorm the need for maximizing packaging. Using a placemat strategy, half the groups answer Question 1 and half the groups answer Question 2.

1) In what contexts is it important to obtain the maximum volume for a rectangular package?
2) In what contexts is the maximum volume package not necessarily the most desirable?

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Action!

Pairs → Activity

Have students work in pairs to complete BLM 4.4B.1. Review some of the functions of the handheld device beforehand, if necessary.

Process Expectations/Observation/Rubric:
Observe and listen to students as they engage in problem solving and reasoning and proving.

Mathematical Process Focus: Reasoning – Students will make logical connections between properties of derivatives and properties of functions.

Consolidate Debrief

Whole Class → Discussion

Have students share their solutions which maximize the volume. Have them reflect on these dimensions with respect to questions 1 and 2 from Minds On. What might be a good application for this box?

Small Groups → Activity

Students begin BLM 4.4B.2. This can be finished for homework or worked on in small groups. Question 2 could be left until next class, and worked through as a whole class.

Practice
Application

Home Activity or Further Classroom Consolidation

Complete BLM 4.4B.2 for next class.

This lesson is modified, with permission, from TI instructional materials.

Detailed instructions for this lesson using TI—Nspire CAS or TI-89s are provided in a separate file. They may be distributed to students if they are new to the technology.

Placemat activity. See pages 30-33 of *Think Literacy: Cross-Curricular Approaches, Grades 7-12* for more information on graphic organizers

4.4B.1: Build That Box!

In this activity, you will use a Computer Algebra System (CAS) to solve a very common calculus problem. You are provided with a sheet of metal that measures 80 cm by 60 cm. If you cut squares of equal sizes out of the corners, you are left with a rectangle in the centre and four flaps that can fold up to form the sides of a box. Find the size of the square that you need to cut from each corner in order to maximize the volume of the box.

Part 1 – Numerical Investigation

The size of the squares that you cut out will also serve as the height of the box. You will need to construct algebraic expressions for the length and the width.

1. In the first column of the lists, store the values {0, 1, 2, 3} as the first few cut sizes. Calculate the length and width of the box for each cut size and copy your values into the table.

Cut size	Length	Width

2. Using the values from your table, perform a regression based upon Length vs. Cut size. What is the algebraic expression for the length?

3. Using the values from your table, perform a regression based upon Width vs. Cut size. What is the algebraic expression for the width?

4. Write an expression for a function that represents the volume of the box as a function of the cut size.

4.4B.1: Build That Box! (cont.)

Part 2 – The Algebraic Investigation

Show the commands that you use to accomplish each of the following. You do not need to list the menus and options that you accessed.

1. Define the function *v(x)* to represent the volume of the box.
2. Define the first derivative function in *v1(x)*.
3. Define the second derivative function in *v2(x)*.
4. Determine the roots of the first derivative function.
5. Use the second derivative function to classify the roots as either a maximum or minimum for *v(x)*. Explain your thinking.
6. Determine the maximum volume of the box. Show your reasoning.

4.4B.1: Build That Box! (cont.)

7. Find the dimensions of the box that yields the maximum volume. Show all your steps.

4.4B.2: More Boxes!

1. Work out the maximum volume for boxes with the following lengths and widths.

a) Length: 64 Width: 30

b) Length: 21.5 Width: 19.5

2. Determine a formula for the “general case” (i.e., where “*x*” is the side length of the square cut out, “*l*” is the length of the sheet and “*w*” is the width).

Unit 4: Day 8: Applications to Business Economics

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Minds On: 5 Action: 50 Consolidate:20 Total=75 min	**Math Learning Goals:** • Solve a variety of optimization problems given an algebraic model • Solve a variety of optimization problems requiring the creation of an algebraic model	**Materials** • Graphing Calculator • BLM 4.8.1 • BLM 4.8.2

Assessment Opportunities

Minds On…

Student Grouping → Instructional Strategy

Discuss the terminology and equations used for cost problems. Have students try to develop the equations. Students can fill in answers on BLM 4.8.1

Discuss what PPC price means.
PPC Price – pay-per-click, an internet marketing formula used to price online advertisements.

Assessment Opportunities: Graphing Calculators can be used to find the vertex of the parabola to save time, and to represent the problem visually.

Action!

Whole Class → Instructional Strategy

Apply the cost definitions to solve Problem 1 and Problem 2.

Small Group → Activity

Students can use graphing calculators, trial and error, or optimization to complete BLM 4.8.1. Ensure that students relate their findings to the graph as they look at different methods to solve the problem. In checking their process and proving that it is correct , students may use calculators and second derivative test to support their answer. Students will use the relationship between the vertex of a parabola and the maximum/minimum points.

Curriculum Expectations/Observations/Anecdotal

Students should have a solid understanding of solving optimization problems. Feedback regarding next steps and areas of strength can be provided.

Mathematical Process/Making Connections/ Reasoning and Proving

Consolidate Debrief

Whole Class → Discussion

Discuss the importance of optimization in the business world and how real life situations will involve more parameters and constraints, making the question more complex. (I.e., Problem #2: the company could be selling in Canada and another country therefore making the question a partial derivative)

Ask students to provide rationale for the steps of solving Optimization Problems.

Application

Home Activity or Further Classroom Consolidation

Complete BLM 4.8.2

4.8.1: Applications to Economics

In this activity, you will explore several applications of derivatives and analyze real-world processes using graphs and properties of functions.

Cost Equations:

Profit =

Revenue =

Cost =

Therefore, the summarized Profit equation is:

Profit =

PPC Price:

Steps for Solving an Optimization Problem:
Step 1: Draw a diagram if applicable.
Step 2: Define the variables clearly.
Step 3: Write an equation linking the variables.
Step 4: Determine if there are any constraints in the variables.
Step 5: Use derivatives to find the max/min.
Step 6: Solve for variable and test if max/min.
Step 7: State conclusion.

Problem 1:

An electronics company in Canada wants to sell its products in Canada. Assume that prices in Canada decline linearly as the company sells more products. Research was conducted and the following equation was determined for the country.

Price in Canada = \$85 - 0.10*x*

The start-up cost for the electronics company is \$20 000 and the manufacturing cost is \$5.00 per item.

a) Determine the number of items that should be sold for the company to maximize the profit.
b) Graph the equation on a graphing calculator after you have determined what value(s) for *x* will optimize the profit. Relate the *x* value that you found by determining the derivative to the graph.

Definitions for the Parameters:
Let *x* represent the number of items sold in Canada.
Let *P* represent the profit for the Canadian market.

4.8.1: Applications to Economics (cont.)

Cost equations in relation to the question are:

Revenue =

Cost =

Profit =

Find the values of x which will maximize P(x).

4.8.1: Applications to Economics (cont.)

Problem 2:
When an electronics company charges $100 for its product it sells 500 units. The company decides to have a promotional sale and sell its product for $75. With the sale the company sells 1200 units. The company also advertises online. The company pays $1.00 per click and typically receives a total of 250 clicks. When the company increased the price per click to $2.50 it received a total of 900 clicks.

a) Determine the maximum price per click for a paid search engine.
b) Graph the equation on the graphing calculator after you have determined what value(s) for x will optimize the profit. Relate the x value that you found to the derivative of the graph.

Definitions for the Parameters:

Product Price Equation:

PPC Price Equation:

Sub into Cost Equations:

Revenue=

Cost =

Profit =

4.8.2: Applications to Economics Homework

1) A hockey team plays in an arena that holds 100,000 spectators. Each ticket is sold at \$15, producing an average attendance of 50,000. When the ticket price is lowered to \$10, the average attendance rises to 65,000.

a) Find an equation to represent the number of tickets sold for the given price, p.

b) How should the ticket prices be set to maximize revenue?

2) The cost price of a Saturn is \$13 000. When the dealer sells each car for \$25 000, he sells 21 cars per month. For each reduction of \$1000 in the selling price, the dealer sells 3 more cars each month.

a) Determine the Profit equation for this function.

b) Explain how the profit function was determined. Explain each variable.

c) Determine the selling price of a car for maximum monthly profit.

3) A farmer wants to put a fence around his horses that is an area of 400 cm^2. The fence will be put up in a rectangular field using fencing material costing \$2.50 per cm, and then divide it in thirds with partitions, constructed from material costing \$1.00 per cm. The partitions are parallel to the side, which gives the width of the field, find the dimensions of the field that will produce the cheapest design.

a) What is the length and width?

b) What is the total cost of the cheapest design?

4) The Better View Company sells computer screens at \$60 each. They give a discount of \$0.10n per computer screen, where n is the number of computer screens purchased. Find the number of computer screens the company should persuade Best Buy to purchase in order to maximize revenue.

5) A storage company owns 150 storage spaces that can be rented at \$40 per month each. For each \$7 month increase in rent there is one vacancy created that cannot be filled. What should the monthly rent be to maximize the total revenue?

6) A company estimates that the cost (in dollars) of producing x items is $C(x) = 2400 + 3x + 0.0001x^2$.

a) Find the average cost of producing 1000 items.

b) At what production level will the average cost be lowest, and what is this minimum average cost?